5th homework
Due date: 4/21

Recall that a matrix $A \in M_n(\mathbf{R})$ is orthogonal if $AA^{\mathrm{t}} = I_n$. and that a matrix $A \in M_n(\mathbf{C})$ is unitary if $AA^* = I_n$ and A is normal if $AA^* = A^*A$. Let $i = \sqrt{-1} \in \mathbf{C}$.

Exercise 1. Let $A, B \in M_n(\mathbf{R})$. Show that $A + iB$ is unitary if and only if $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in M_{2n}(\mathbf{R})$ is orthogonal.

Exercise 2. Show that $A \in M_n(\mathbf{C})$ is normal if and only if there exists a polynomial $Q \in \mathbf{C}[X]$ such that $A^* = Q(A)$.

Exercise 3. Let

$$A = \begin{pmatrix} 1 & i & 1 \\ -i & 1 & i \\ 1 & -i & 1 \end{pmatrix}.$$

Find an invertible matrix $P \in M_3(\mathbf{C})$ such that P is unitary and P^*AP is diagonal.

Exercise 4. Let V be a finite dimensional inner product space over $\mathbf{C}$. Let $T : V \to V$ be a self-adjoint operator. Show that

(1) $1 + iT$ is invertible, and then
(2) $S := (1 - iT)(1 + iT)^{-1}$ is a unitary operator.

Exercise 5. Let $A = (a_{i,j}) \in M_n(\mathbf{C})$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{C}$ be the eigenvalues of A (counted with multiplicity). Show that A is normal if and only if

$$\sum_{1 \le i,j \le n} |a_{i,j}|^2 = \sum_{k=1}^{n} |\lambda_k|^2.$$