4th homework
Due date: 3/31

Let $(V, \langle\ ,\ \rangle)$ be a inner product space over F ($F = \mathbf{R}$ or $\mathbf{C}$).

Exercise 1. Let W be a subspace of V. Suppose the orthogonal projection $T := \mathrm{Proj}_W : V \to V$ onto W exists.

(1) Show that T is self-adjoint.
(2) Find a self-adjoint operator S such that $S^2 = T + 1$.

Exercise 2. Suppose that V is finite dimensional. Let $T : V \to V$ be a normal operator.

(1) Prove that $\mathrm{Ker}\, T = \mathrm{Ker}\, T^n$ for any $n \geq 1$.
(2) $\mathrm{Im}\, T = \mathrm{Im}\, T^n$ for any $n \geq 1$.

Exercise 3. Let $A \in M_3(\mathbf{R})$ with $AA^{\mathrm{t}} = 1$ and $\det A = 1$. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by $T(v) = A \cdot v$. Show that there exists a line L passing through the origin such that for any $v \in \mathbf{R}^3$, $T(v)$ is obtained by rotating v around L by an angle θ.

Exercise 4. For any linear transformation $T : V \to V$, define a new pairing $\langle x, y\rangle_T := \langle Tx, y\rangle$ for $x, y \in V$. We say T is *positive* if $\langle\ ,\ \rangle_T$ is a inner product on V.

(1) Show that T is positive if and only if T is self-adjoint and all eigenvalues of T are positive real numbers.
(2) If S and T are two positive linear transformations, show that ST is positive if and only if S and T commute, i.e. $ST = TS$.

Exercise 5. Let

$$A = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix} \in M_3(\mathbf{R}).$$

For $x, y \in \mathbf{R}^3$ (viewed as column vectors), define

$$\langle x, y\rangle_A := x^{\mathrm{t}} A y.$$

Find the range of a such that $\langle\ ,\ \rangle_A$ is a inner product on $\mathbf{R}^3$.

Exercise 6. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Find $P \in M_4(\mathbf{R})$ such that $PP^{\mathrm{t}} = I_4$ and $P^{-1}AP$ is a diagonal matrix.