2nd homework
Due date: 3/17

There are six problems.

Exercise 1. Find a set of three polynomials $p_1(x) = a$, $p_2(x) = b + cx$ and $p_3(x) = d + ex + fx^2$ with $a, b, c, d, e, f \in \mathbf{R}$ such that $\{p_1(x), p_2(x), p_3(x)\}$ is an orthonormal set with respect to the inner product $\langle f, g\rangle = \int_0^2 f(x)g(x)dx$.

Exercise 2. Let $(V, \langle\ ,\ \rangle)$ be a inner product space over $\mathbf{C}$. For each $x, y \in V$ show that

$$\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2;$$

$$4\langle x, y\rangle = \sum_{k=0}^{3} i^k \|x + i^k y\|^2.$$

(recall that $\|x\| := \sqrt{\langle x, x\rangle}$).

Exercise 3. Let $V = \mathbf{R}[x]$ be the space of polynomials with coefficients in $\mathbf{R}$. Let $b > a$ be real numbers. Define the inner product on V by

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx.$$

For each positive integer n, define

$$q_{2n}(x) = (x - a)^n (x - b)^n,$$
$$p_n(x) = \frac{d^n}{dx^n}(q_{2n}(x)).$$

(1) Show that
$$\frac{d^{i-1}q_{2n}}{dx^{i-1}}(a) = \frac{d^{i-1}q_{2n}}{dx^{i-1}}(b) = 0$$
for all $i = 1, 2, , \ldots, n$.
(2) Show that p_n has degree n.
(3) Show that $p_1, p_2, \ldots, p_n$ are orthogonal(or perpendicular) to each other.

Exercise 4. Let $\langle\ ,\ \rangle$ be the standard inner product on $\mathbf{R}^3$ given by $\langle(a_1, a_2, a_3), (b_1, b_2, b_3)\rangle = a_1b_1 + a_2b_2 + a_3b_3$. Let $v_1 = (1, 0, 1)$, $v_2 = (1, 0, -1)$ and $v_3 = (0, 3, 4)$. Apply the Gram-Schmidt process to $\{v_1, v_2, v_3\}$ to obtain an orthonormal set $\{w_1, w_2, w_3\}$.

Exercise 5. Let
$$\Omega := \begin{pmatrix} 6 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \in M_3(\mathbf{R}).$$
Find $A \in M_3(\mathbf{R})$ such that $AA^* = \Omega$ (so Ω is positive definite).

(Hint: Use Gram-Schmidt process).

Exercise 6. Let $(V, \langle\ ,\ \rangle)$ be an inner product space. If $T : V \to V$ is a projection (i.e. $T^2 = T$) such that $\|T(x)\| \leq \|x\|$ for all $x \in V$, show that T is an orthogonal projection.

(Hint: You may need to show $\langle x, y\rangle = 0$ for all $x \in \operatorname{Ker} T$ and $y \in \operatorname{Im} T$.)