1st homework
Due date: 3/3

Let F be either $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$ and let V be a finite dimensional vector space over F. Let n be a positive integer.

Exercise 1. Let $V = F^4$. Let $v_1 = (2,1,2,3)$ and $v_2 = (1,-1,0,2)$ be vectors in V. Let $W = Fv_1 + Fv_2$ be a subspace of V. Let $v_3 = (1,1,0,0)$ and $v_4 = (0,2,2,-1)$. Show that $\{[v_3],[v_4]\}$ is a basis of V/W. Let $v_5 = (1,2,3,-2)$. Then

$$[v_5] = \alpha_1 \cdot [v_3] + \alpha_2 \cdot [v_4]$$

for $\alpha_1, \alpha_2 \in F$. Find α_1, α_2.

Exercise 2. Let V_1 and V_2 be subspaces of V. Let $i : V_1 \hookrightarrow V_1 + V_2$ be the inclusion map. Show that the inclusion map i induces an isomorphism

$$\bar{i} : \frac{V_1}{V_1 \cap V_2} \simeq \frac{V_1 + V_2}{V_2},$$

where $\bar{i}$ is the map defined by

$$\bar{i}(v + V_1 \cap V_2) := i(v) + V_2.$$

Exercise 3. Let $T : V \to V$ be a linear transformation. Let $W \subset V$ be a T-invariant subspace. Show that T is an isomorphism if and only if $T : W \to W$ and $\overline{T} : V/W \to V/W$ are both isomorphisms.

Exercise 4. Let m be any positive integer. Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m \in V$ be non-zero vectors in V. Show that there exists a linear functional $\ell \in V^\vee$ such that

$$\ell(\alpha_i) \neq 0 \text{ for all } i = 1, 2, \ldots, m.$$

Exercise 5. If $\ell \in \mathrm{Hom}_F(M_n(F), F)$ is a linear functional of $M_n(F)$, show that there exists $B \in M_n(F)$ such that

$$\ell(A) = \mathrm{Tr}(AB) \text{ for all } A \in M_n(F).$$

Exercise 6. Let $\ell : M_n(F) \to F$ be a linear functional. If $\ell(AB) = \ell(BA)$ for all $A, B \in M_n(F)$, show that ℓ is a multiple of the trace function, i.e. there exists $\alpha \in F$ such that $\ell(A) = \alpha \cdot \mathrm{Tr}(A)$ for all $A \in M_n(F)$.