12th homework
Due date: 12/16

Let F be either $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$ and let V be a finite dimensional vector space over F. Recall that for $v \in V$, let

$$C_v(f) := \operatorname{span}_F \left\{v, Tv, T^2v, \ldots, T^{n-1}v\right\}$$

the smallest T-invariant subspace containing v with f the characteristic polynomial of T restricted to $C_v(f)$.

Exercise 1. Let $V = \mathbf{Q}^4$ and

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 1 \\ -1 & 3 & -1 & 0 \end{pmatrix} \in M_4(\mathbf{Q}).$$

Let $T\colon V \to V$ be the linear transform defined by $T(v) := A \cdot v$. Decompose V into a direct sum of T-cyclic subspaces

$$V = \bigoplus_{i=1}^{r} C_{v_i}(f_i^{n_i}),$$

where $f_i \in \mathbf{Q}[x]$ are irreducible polynomials. Determine the set of irreducible polynomials $\{f_1, f_2, \ldots, f_r\}$ and a set $\{v_1, v_2, \ldots, v_r\}$ of generators.

Exercise 2. Let $T : V \to V$ be a linear transformation. Show that if V is T-cyclic, then every linear transformation $S : V \to V$ that commutes with T is a polynomial in T.

Exercise 3. Describe all linear transfomation $T : V \to V$ such that V has only finitely many T-invariant subspaces.

Exercise 4. Let $A, B \in M_n(\mathbf{C})$ and let $T : M_n(\mathbf{C}) \to M_n(\mathbf{C})$ be the linear transformation defined by

$$T(X) = AX - XB, \quad X \in M_n(\mathbf{C}).$$

Prove that the eigenvalues of A are distinct from eigenvalues of B if and only if T is an isomorphism.