1st homework
Due date: 9/30

Exercise 1. Let $v_1 = (1, 2, , 3)$ and $v_2 = (1, -1, 1)$ be two vectors in the vector space $\mathbf{R}^3$ over $\mathbf{R}$. Determine if $(5, 1, 9)$ is a linear combination of v_1 and v_2 (or equivalently if $(5, 1, 9)$ belongs to the vector space $\operatorname{span}_{\mathbf{R}} \{v_1, v_2\}$.

Exercise 2. Let

$$S = \left\{x^3 + x^2 + x + 1, x^2 + x + 1, x + 1\right\} \subset \mathbf{R}[x]$$

be a subset of the vector space $\mathbf{R}[x]$ over $\mathbf{R}$. Determine if $-x^3 + 2x^2 + 3x + 3$ and $2x^3 - x^2 + x + 3$ are linear combinations of vectors in S. Justify your answer.

Exercise 3. Let V be the set of all functions $f : \mathbf{R} \to \mathbf{R}$. Then V is a vector space over $\mathbf{R}$ endowed with the usual addition and scalar product between real-valued functions. Let

$$S = \left\{1, x, x^2, \ldots, x^n, \ldots\right\}$$

be the subset of monomials in V. Show that the function $f(x) = \sin x$ is NOT a linear combination of finitely many vectors in S over $\mathbf{R}$.

Exercise 4. Let V be a vector space over $\mathbf{R}$. Show that if a subset $\{v_1, v_2, \ldots, v_n\}$ of V is linearly independent over $\mathbf{R}$, then so is the set $\{v_1 - 2v_2, v_2 - 2v_3, \ldots, v_{n-1} - 2v_n, v_n\}$.

Exercise 5. Let $V = \mathbf{R}^2$. Define the addition by

$$(x_1, x_2) \boxplus (y_1, y_2) = (x_1 + y_1, x_2 y_2),$$

and define the scalar product by

$$\alpha \boxdot (x_1, x_2) := (\alpha x_1, x_2).$$

Verify if V is a vector space over $\mathbf{R}$ with $\boxplus$ and $\boxdot$.

Exercise 6. Let $V = \mathbf{R}^2$. Then V is equipped with the usual addition $+$. Define a scalar product by

$$\alpha \boxdot (x_1, x_2) := (\alpha x_1, -\alpha x_2), \quad \alpha, x_1, x_2 \in \mathbf{R}.$$

Verify if V is a vector space over $\mathbf{R}$ with the usual $+$ and $\boxdot$.

Exercise 7. Let V be a vector space over $\mathbf{R}$ and let W_1, W_2 and W_3 be subspaces of V. Suppose that

$$W_3 \subset W_1 \cup W_2.$$

Show that either $W_3 \subset W_1$ or $W_3 \subset W_2$.