Perfection for strong edge-coloring on graphs

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Abstract

A strong edge-coloring of a graph is a function that assigns to each edge a color such that every two distinct edges that are adjacent or adjacent to a same edge receive different colors. The strong chromatic index $\chi'_s(G)$ of a graph $G$ is the minimum number of colors used in a strong edge-coloring of $G$. From a primal-dual point of view, there are three natural lower bounds of $\chi'_s(G)$, that is $\sigma(G) \leq \sigma^*(G) \leq am(G) \leq \chi'_s(G)$. For any $t \in \{\sigma, \sigma^*, am\}$, a graph $G$ is vertex $t$-perfect (respectively, edge $t$-perfect) if $t(H) = \chi'_s(H)$ for any induced (respectively, edge-induced) subgraph $H$ of $G$. The aim of this paper is to study the above versions of perfection on strong edge-coloring.

1 Introduction

The purpose of this paper is to study perfection on strong edge-coloring. A strong edge-coloring of a graph is a function that assigns to each edge a color such that every two distinct edges that are adjacent or adjacent to a same edge receive different colors. The strong chromatic index $\chi'_s(G)$ of a graph $G$ is the minimum number of colors used

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in a strong edge-coloring. An induced matching of a graph is a set of edges in which every two distinct edges are not adjacent and not adjacent to a same edge. Notice that a color class of a strong edge-coloring is an induced matching.

From a primal-dual point of view, there are three natural lower bounds on the strong chromatic index. An anti-matching is a set of edges in which every two distinct edges are adjacent or are adjacent to a same edge. The anti-matching number $am(G)$ of a graph $G$ is the maximum size of an anti-matching. The closed edge-neighborhood of a clique $C$ is the set $N_e[C] = \{e \in E: e$ is incident to some vertex in $C\}$. The closed edge-neighborhood of an edge $xy$ is the set $N_e[xy] = \{e \in E: e$ is incident to $x$ or $y\}$. Denote by $\sigma^*(G)$ (respectively, $\sigma(G)$) the maximum size of a closed edge-neighborhood of a clique (respectively, an edge). In other words, for a graph $G = (V, E)$,

$$\sigma^*(G) := \max_{C: \text{clique}} |N_e[C]| = \max_{C: \text{clique}} \left( \sum_{x \in C} d_G(x) - \binom{|C|}{2} \right)$$

and

$$\sigma(G) := \max_{xy \in E} |N_e[xy]| = \max_{xy \in E} (d_G(x) + d_G(y) - 1).$$

Since an edge is a clique, a closed edge-neighborhood of a clique is an anti-matching and an anti-matching has at most one edge in common with an induced matching, we have the following weak duality inequalities that for any graph $G$,

$$\sigma(G) \leq \sigma^*(G) \leq am(G) \leq \chi'_s(G). \quad (1)$$

Faudree, Gyárfás, Schelp and Tuza [12] showed that $\sigma(Q^d) = \sigma^*(Q^d) = 2d - 1 < 2d = am(Q^d) = \chi'_s(Q^d)$, where $Q^d$ is the $d$-dimensional cube whose vertices are the $d$-tuples of $\{0, 1\}$ entries and whose edges are the pairs of $d$-tuples with exactly one different position. For any $t \in \{\sigma, \sigma^*, am\}$, a graph $G$ is vertex $t$-perfect (respectively, edge $t$-perfect) if $t(H) = \chi'_s(H)$ for any induced (respectively, edge-induced) subgraph $H$ of $G$.

Strong edge-coloring was first studied by Fouquet and Jolivet [13, 14] for cubic planar graphs. By a greedy algorithm, it is easy to see that $\chi'_s(G) \leq 2\Delta^2 - 2\Delta + 1$ for any graph $G$ of maximum degree $\Delta$. Fouquet and Jolivet [13] established a Brooks type upper bound $\chi'_s(G) \leq 2\Delta^2 - 2\Delta$, which is not true only for $G = C_5$ as pointed out by Shiu and Tam [28]. Conjecture 1 below was posed by Erdős and Nešetřil [10, 11] and revised by Faudree, Gyárfás, Schelp and Tuza [12] who also gave a weaker conjecture.

**Conjecture 1.** ([10, 11, 12]) *If $G$ is a graph of maximum degree $\Delta$, then $\chi'_s(G) \leq \Delta^2 + \lfloor \frac{\Delta}{2} \rfloor^2$.***
Conjecture 2. ([12]) If \( G \) is a graph of maximum degree \( \Delta \), then \( \text{am}(G) \leq \Delta^2 + \lfloor \frac{\Delta}{2} \rfloor^2 \).

For graphs with maximum degree \( \Delta = 3 \), Conjecture 1 was verified by Andersen [1] and by Horák, Qing and Trotter [17] independently. For \( \Delta = 4 \), while Conjecture 1 says that \( \chi'_s(G) \leq 20 \), Horák [16] obtained \( \chi'_s(G) \leq 23 \) and Cranston [9] proved \( \chi'_s(G) \leq 22 \). Faudree, Gyárfás, Schelp and Tuza [12] proved that there exists a constant \( \epsilon > 0 \) such that \( \text{am}(G) \leq (2 - \epsilon)\Delta^2 \) for any graph \( G \) of maximum degree \( \Delta \). Molloy and Reed [24] proved that for large \( \Delta \), every graph of maximum degree \( \Delta \) has \( \chi'_s(G) \leq 1.998\Delta^2 \) using a probabilistic method. Mahdian [21] proved that \( \chi'_s(G) \leq (2 + o(1))\Delta^2 / \ln \Delta \) for a \( C_4 \)-free graph \( G \). Faudree, Gyárfás, Schelp and Tuza [12] proved that \( \chi'_s(G) \leq \Delta^2 \) for graphs in which all cycle lengths are multiples of four. They mentioned that this result probably could be improved to a linear function of the maximum degree. Brualdi and Massey [2] improved the upper bound to \( \chi'_s(G) \leq \alpha \beta \) for such graphs, where \( \alpha \) and \( \beta \) are the maximum degrees of the respective partitions. Nakprasit [25] proved that if \( G \) is bipartite and the maximum degree of one partite set is at most 2, then \( \chi'_s(G) \leq 2\Delta \). Chang and Narayanan [8] proved that \( \chi'_s(G) \leq 8\Delta - 6 \) for chordless graphs \( G \). This settles the above question by Faudree, Gyárfás, Schelp and Tuza [12] in the positive, since graphs with cycle lengths divisible by 4 are chordless graphs. They [8] also established that \( \chi'_s(G) \leq 10\Delta - 10 \) for 2-degenerate graphs \( G \).

Strong edge-coloring on planar graphs also extensively studied in the literature. Faudree, Schelp, Gyárfás and Tuza [12] asked whether \( \chi'_s(G) \leq 9 \) if \( G \) is cubic planar. If this upper bound is proved to be true, it would be the best possible. Faudree, Gyárfás, Schelp and Tuza [12] showed that \( \chi'_s(G) \leq 4\Delta(G) + 4 \) for any planar graph \( G \) of maximum degree \( \Delta \) by the four-color theorem. They also exhibited a planar graph \( G \) whose strong chromatic index is \( 4\Delta(G) - 4 \). Their proof also gives a consequence that \( \chi'_s(G) \leq 3\Delta \) for planar graphs \( G \) of girth at least 7. Chang, Montassier, Pecher and Raspaud [7] further proved that \( \chi'_s(G) \leq 2\Delta - 1 \) for planar graphs \( G \) with large girth. Strong chromatic index for Halin graphs was first considered by Shiu, Lam and Tam [27] and then studied in [6, 18, 20, 28]. For other results on strong edge-coloring, see [3, 15, 22, 23, 26, 30].

The aim of this paper is to study the above mentioned perfection on strong edge-coloring. Notice that edge \( t \)-perfection implies vertex \( t \)-perfection for \( t \in \{\sigma, \sigma^*, \text{am}\} \). Also, \( \sigma \)-perfection implies \( \sigma^* \)-perfection, which in turn implies am-perfection. See the
following flowchart for a summary.

\[
\text{edge } \sigma\text{-perfect } \Rightarrow \text{edge } \sigma^*\text{-perfect } \Rightarrow \text{edge am-perfect} \\
\downarrow \quad \downarrow \quad \downarrow
\text{vertex } \sigma\text{-perfect } \Rightarrow \text{vertex } \sigma^*\text{-perfect } \Rightarrow \text{vertex am-perfect}
\]

2 Preliminary

For an integer \( n \geq 3 \), the \( n \)-cycle is the graph \( C_n \) with vertex set \( V(C_n) = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n\} \), where \( v_{n+1} = v_1 \). Graph \( C(n, d) \) is the graph obtained from \( C_n \) by adding \( d \) new vertices adjacent to each vertex of \( C_n \). A block graph is a graph whose blocks are complete graphs. A cactus is a graph whose blocks are cycles or complete graphs of two vertices. Notice that cacti are planar graphs and include trees. Also, graph \( C(n, d) \) is a cactus.

Lemma 3. ([5]) If \( H \) is a subgraph of \( G \), then \( \chi'(H) \leq \chi'_s(G) \).

Lemma 4. ([5]) If integer \( n \geq 3 \), then \( \chi'_s(C_n) = 5 \) for \( n = 5 \), \( \chi'_s(C_n) = 3 \) for \( n \) a multiple of \( 3 \) and \( \chi'_s(C_n) = 4 \) otherwise.

Lemma 5. ([5]) If \( G = C(n, d) \) with \( n \geq 3 \) and \( d \geq 1 \), then

\[
\chi'_s(G) = \begin{cases} 
\sigma(G) + 1, & \text{if } n = 4 \text{ or } (n, d) = (7, 1); \\
\sigma(G), & \text{if } n \geq 6 \text{ is even}; \\
\lceil \frac{2n(d+1)}{n-1} \rceil, & \text{if } n \geq 3 \text{ is odd but } (n, d) \neq (7, 1).
\end{cases}
\]

Liao [19] established the following result for cacti.

Theorem 6. ([19]) If \( G \) is a cactus in which the length of a cycle is a multiple of \( 6 \), then \( \chi'_s(G) = \sigma(G) \).

Corollary 7. ([12]) If \( T \) is a tree, then \( \chi'_s(T) = \sigma(T) \).

Corollary 8. If \( G \) is a cactus in which the length of a cycle is a multiple of \( 6 \), then \( G \) is edge \( t \)-perfect and vertex \( t \)-perfect for \( t \in \{\sigma, \sigma^*, \text{am}\} \).

A graph \( G \) is chordal if it has no induced cycle of length at least four in \( G \). Cameron [3] proved that \( \chi'_s(G) = \sigma^*(G) \) for any chordal graph \( G \). We now give an algorithm for strong edge-coloring on chordal graphs, which provides an alternative
proof of the same result. It is well-known that a chordal graph \( G = (V, E) \) has a \textit{perfect elimination scheme}, which is an ordering \( v_1, v_2, \ldots, v_n \) of \( V \) such that
\[
i < j < k, v_i v_j \in E \text{ and } v_i v_k \in E \text{ imply } v_j v_k \in E. \quad \text{(PEO)}
\]
In other words, the set \( C_i = \{ v_i \} \cup \{ v_j : i < j, v_i v_j \in E \} \) is a clique for \( 1 \leq i \leq n \).

The algorithm for strong edge-coloring on chordal graphs is a greedy one as follows.

1. initially all edges are un-colored;
2. \textbf{for} \( j = n \) to \( 1 \) \textbf{step by} \(-1\)
3. \textbf{for} (any un-colored edge \( v_i v_j \) incident to \( v_j \)) \textbf{do}
4. color \( v_i v_j \) by the least positive integer not used by an edge \( v_p v_q \) such that \( \{ v_p v_q, v_i v_j \} \) is an anti-matching;
5. \textbf{end for}
6. \textbf{end for}

Suppose \( s \) colors are used, and edge \( v_i v_j \) is colored by \( s \) at iteration \( j \). By the back do loop, \( i < j \). First, by the coloring method, the edge-coloring is strong and so \( \chi'_s(G) \leq s \). For any \( 1 \leq r < s \), there is an edge \( v_p v_q \) with \( p < q \) that was colored by \( r \) before \( v_i v_j \) being colored such that \( \{ v_p v_q, v_i v_j \} \) is an anti-matching. Then \( i < j \leq q \) by the back do loop. We consider 4 cases.

Case 1. \( j = q \) or \( j < q \) with \( v_j v_q \in E \). In this case, \( v_q \in C_j \) and so \( v_p v_q \in N_e[C_j] \).

Case 2. \( j > p \) with \( v_j v_p \in E \) or \( j = p \) or \( j < p \) with \( v_j v_p \in E \). For the first subcase, by (PEO), we are back to Case 1. For the other two subcases, \( v_p \in C_j \) and so \( v_p v_q \in N_e[C_j] \).

Case 3. \( i < q \) with \( v_i v_q \in E \). In this case, by (PEO), we are back to Case 1.

Case 4. \( i > p \) with \( v_i v_p \in E \) or \( i = p \) or \( i < p \) with \( v_i v_p \in E \). For the first two subcases, by (PEO), we are back to Case 3. For the last subcase, by (PEO), we are back to Case 2.

So, in any case, \( N_e[C_j] \) contains one edge colored by \( r \) for \( 1 \leq r \leq s \). Hence, \( s \leq |N_e[C_j]| \leq \sigma^*(G) \geq \chi'_s(G) \leq s \), giving that \( \sigma^*(G) = \chi'_s(G) = s \). This not only gives an optimal strong edge-coloring of \( G \) but also gives a proof for Theorem 9.

\textbf{Theorem 9. ([3])} If \( G \) is chordal, then \( \chi'_s(G) = \sigma^*(G) \).

\textbf{Corollary 10.} Chordal graphs are vertex \( \sigma^* \)-perfect and vertex am-perfect.

\textbf{Lemma 11.} If \( G \) has no clique of size 3 in which each vertex is of degree at least 3, then \( \sigma(G) = \sigma^*(G) \).
Proof. Suppose to the contrary that \( \max_{xy \in E} |N_e[xy]| = \sigma(G) < \sigma^*(G) = \max_{C:\text{clique}} |N_e[C]|. \)
Then the clique \( C^* \) attaining the maximum of the last equality has at least 3 vertices. By the assumption, all except 2 vertices \( x^* \) and \( y^* \) of \( C^* \) are of degree 2. Hence \( |N_e[x^*y^*]| = N_e[C^*] \), which is impossible. This completes the proof of the lemma. \( \square \)

**Corollary 12.** Chordal graphs without any clique of size 3 in which each vertex is of degree at least 3 are vertex \( t \)-perfect for \( t \in \{\sigma, \sigma^*, \text{am}\} \).

A graph \( G \) is weakly chordal if neither the graph nor its complement contains an induced cycle of length at least five in \( G \).

**Theorem 13.** ([4]) If \( G \) is weakly chordal, then \( \chi'_s(G) = \text{am}(G) \).

**Corollary 14.** Weakly chordal graphs are vertex \( \text{am} \)-perfect.

### 3 Graphs with cycle lengths of multiple 3

By definition and Lemma 4, the length of a cycle in a vertex \( \sigma \)-perfect graph is a multiple of 3. However, the converse is not necessarily true. Notice that by Lemma 5, if \( n \) is odd with \( n < 2d + 3 \) then \( \chi'_s(C(n, d)) > \sigma(C(n, d)) \). On the other hand, \( \chi'_s(C(n, d)) = \sigma(C(n, d)) \) if \( n \geq 6 \) is even and \( d \geq 1 \).

By Corollary 8, a cactus in which the length of a cycle is a multiple of 6 is vertex \( \sigma \)-perfect. However, the condition that a graph with all cycles are of length multiple of 6 is not necessary gives that \( \chi'_s(G) = \sigma(G) \) as shown in the following example. The graph \( G \) in Figure 1 has 15 edges and \( \sigma(G) = 5 \). Suppose \( \chi'_s(G) = 5 \). Since an induced matching in \( G \) has at most 3 edges, each color class has exactly 3 edges. However each color class containing an edge incident to \( x \) must contain two pendent edges whose distance is three from \( x \), a contradiction.

![Figure 1: \( \chi'_s(G) > \sigma(G) \).](image)

In the following we prove that \( \chi'_s(G) = \sigma(G) \) for any 2-connected graph \( G \) in which every cycle has a length a multiple of 3. To establish this result, we need a
useful lemma as follows. In fact, we suspect that these kind of graphs are vertex \( \sigma \)-perfect, although a proof is still not available.

**Lemma 15.** Suppose \( H = (X, Y, E) \) is a bipartite graph in which any vertex in \( Y \) is of degree at most two. Suppose every vertex \( x \in X \) has a list \( L(x) \) of size \( d(x) \) such that \( H \) has no special component that is a path \( x_0, y_1, x_1, y_2, x_2, \ldots, y_r, x_r \) such that \( L(x_0) = L(x_r) \subseteq L(x_i) \) for \( 0 \leq i \leq r \). Then \( H \) has a proper edge-coloring \( f \) such that \( f(xy) \in L(x) \) for any edge \( xy \) with \( x \in X \).

**Proof.** We shall prove the lemma by induction on the number of edges. The case of \(|E| = 1\) is clear. Suppose \(|E| \geq 2\).

**Case 1.** \( H \) has a vertex \( y \in Y \) of degree 1, say \( y \) is adjacent to \( x \in X \).

If \( L(x) = \{c\} \), then we may color \( xy \) by \( c \) and reduce the graph to one with fewer edges that also has no special component. If \(|L(x)| \geq 2\), then we may properly choose a color from \( L(x) \) to color \( xy \) and reduce the graph to one with few edges that also has no special component.

**Case 2.** All vertices in \( Y \) are of degree 2.

If there is a vertex \( x \in X \) of degree 1 with \( L(x) = \{c\} \), say \( x \) is adjacent to \( y \in Y \) whose another neighbor in \( X \) is \( x' \). For the case when \( x' \) is of degree 1 with \( L'(x) = \{c'\} \), as \( H \) has no special component \( c \neq c' \). Then we may color \( xy \) by \( c \) and \( x'y \) by \( c' \) to reduce the graph to one with fewer edges that also has no special component. For the case when \( x' \) is of degree at least two, we may color \( xy \) by \( c \) and properly choose a color \( c' \in L(x') - L(x) \) to color \( x'y' \). This reduces the graph to one with fewer edges that also has no special component, unless there is a special component in the reduced graph \( H' \) with \( x' \) as an end vertex and \( L'(x') = \{c\} \) which extend to a special component in the original graph \( H \) by adding \( y \) and \( x \).

Now suppose all vertices in \( X \) are of degree at least 2. If there is a vertex \( x \in X \) of degree at least 3, then choose a neighbor \( y \) of \( x \) and the other neighbor \( x' \) of \( y \). Choose a color \( c \) in \( L(x) \) to color \( xy \) and a color in \( L(x') - \{c\} \) to color \( x'y \). This reduces the graph to one with fewer edges that also has no special component, as there is at most one vertex of degree one. If all vertices of \( X \) are of degree two, then choose a cycle in the graph. The graph is of even length. For the case when \( L(x) = \{a, b\} \) for all vertices \( x \in X \) in the cycle, it is easy to color the edges in this cycle to reduce the graph to one with few edges that also has no special component. For the case when the cycle has a vertex \( y \in Y \) whose neighbor \( x \) and \( x' \) have \( L(x) \neq L(x') \), we may properly choose
$c \in L(x)$ and $c' \in L(x')$ such that $c \neq c'$ and $L(x) - \{c\} \neq L(x') - \{c'\}$. Coloring $xy$ by $c$ and $x'y$ with $c'$ reduces the graph to one with few edges that also has no special component, as only two vertices $x$ and $x'$ are of degree one but $L(x) \neq L(x')$.

**Theorem 16.** If $G$ is a 2-connected graph in which the length of every cycle is a multiple of 3, then $\chi'_s(G) = \sigma(G)$.

**Proof.** Suppose to the contrary that the theorem is not true. Choose a minimum counterexample $G$ that is a 2-connected graph in which the length of every cycle is a multiple of 3 but $\chi'_s(G) > \sigma(G)$. By Lemma 4, $G$ is not a cycle. So the set $M(G)$ (or $M$ if there is no ambiguity) of vertices of degree at least 3 in $G$ is not empty.

An $M$-path is a path $v_0, v_1, \ldots, v_k$ with $v_0, v_k \in M$ but all other vertices are not in $M$. For any subset $S \subseteq M$, the graph $G_S$ is the subgraph of $G$ induced by all vertices of $G$ in all $M$-paths whose end vertices are in $S$. An $M$-3-block is a nontrivial maximal subgraph $G_S$ for a subset $S \subseteq M$ in which any $M$-path is of length $3\ell$ for some $\ell$. For the graph in Figure 2, $M = \{1, 2, 3, 4, 5, 6\}$ and there are three $M$-3-blocks each is a 6-cycle.

![Figure 2: A graph with three M-3-blocks B₁, B₂ and B₃.](image)

**Claim 1.** There is an $M$-path of length a multiple of 3, and so $G$ has at least one $M$-3-block.

**Proof.** Recall that an ear of a graph $H$ is a path in $H$ that is contained in a cycle and is maximal with respect to the property that internal vertices having degree two in $H$. Since $G$ is 2-connected, it is well known [29] that $G$ has an ear decomposition $P_0, P_1, \ldots, P_r$, which is a decomposition of edges of $G$ such that $P_0$ is a cycle and $P_i$ is an ear of $P_0 \cup P_1 \cup \ldots \cup P_i$ for all $i \geq 1$. Notice that $r \geq 1$ and $P_r$ is an $M$-path from $x$ to $y$. Since $G' = P_0 \cup P_1 \cup \ldots \cup P_{r-1}$ is 2-connected, there are two internally disjoint paths $Q_1$ and $Q_2$ from $x$ to $y$ in $G'$. Any two paths of $Q_1, Q_2$ and $P_r$ form a cycle of length a multiple of 3. Hence these paths are all of length multiple of 3. Therefore, $G$ has at least one $M$-3-block containing the $M$-path $P_r$. □
Now suppose $G$ has $k$ $M$-3-blocks $G_{S1}, G_{S2}, \ldots, G_{Sk}$ for some $k \geq 1$. The $s$-subdivision $S(H, s)$ of a multi-graph $H$ is the graph obtained from $H$ by replacing each edge by a path of length $s$. Notice that an $M$-3-block $G_S$ is the 3-subdivision of a multi-graph whose vertices are those vertices having distance a multiple of 3 from vertices of $S$ in $G_S$.

**Claim 2.** $G \neq G_{S1}$.

**Proof.** Suppose to the contrary that $G = G_{S1}$. Then $G = S(I, 3)$ for some multigraph $I$. Notice that $G$ and $H = S(I, 2)$ have the same maximum degree $\Delta = \Delta(I)$ and $\sigma(G) = \Delta + 1$. Also, $H$ is a bipartite graph with one partite set $X = V(I)$ and the other partite set $Y = V(H) - X$ in which each vertex is of degree 2. Consider the list $L(x) = \{1, 2, \ldots, d_H(x)\}$ for all $x \in X$, where $d_G(x) = d_H(x)$. Since $G$ is 2-connected, $H$ has no special component. By Lemma 15, $H$ has a proper edge-coloring $f$ using colors in $\{1, 2, \ldots, \Delta\}$. We may consider $G = S(I, 3)$ as a graph obtained from $H = S(I, 2)$ by inserting an edge between two edges incident to a vertex of degree two in $H$. Then we may extend the edge-coloring $f$ of $H$ to a strong edge-coloring $g$ of $G$ by coloring all such newly inserted edges by $\Delta + 1$. This gives that $\chi'_s(G) \leq \Delta + 1 = \sigma(G)$. Hence $\chi'_s(G) = \sigma(G)$, a contradiction to the assumption that $\chi'_s(G) > \sigma(G)$. □

Let $G^*$ be the graph obtained from $G$ by contracting each $M$-3-block $G_{Si}$ to a vertex $v_i$ for $1 \leq i \leq k$. That is, the vertices in $S_i$ are replaced by a vertex $v_i$, and any edge between $x \in S_i$ and $y \in S_j$ (respectively, $y$ not in any $S_j$) is replaced by an edge $v_iv_j$ (respectively, $v_iy$).

**Claim 3.** $G^*$ has no loop and no parallel edges.

**Proof.** Suppose to the contrary that $G^*$ has a loop $e$ at some $v_i$. That is, $e$ has two end vertices $x, y \in S_i$ in $G$, and there is a path $P$ of length $3\ell$ between them. Consequently, $P$ together with $e$ form a cycle of length $3\ell + 1$ in $G$, which is impossible.

Suppose to the contrary that $G^*$ has two parallel edges $e_1$ and $e_2$. Then each $e_r$ is between some $v_i$ and some $v_j$ (respectively, some $y$ not in any $S_j$). In $G$, each $e_r$ has one end vertex $x_r \in S_i$, and the other vertex $y_r \in S_j$ (respectively, $y_r = y$). In $G$, there is a path $P_i$ of length $3\ell_i$ from $x_1$ to $x_2$, and a path $P_j$ of length $3\ell_j$ from $y_2$ to $y_1$. Consequently, $P_i, e_2, P_j, e_1$ form a cycle of length $3\ell_1 + 3\ell_2 + 2$ in $G$, which is impossible. □

By Claims 2 and 3, $G^*$ is a simple graph with at least two vertices. In fact, $G^*$
is 2-edge-connected and every cycle in $G$ has a length a multiple of 3. It is then the case that any block $B$ with at most one cut-vertex in $G^*$ has a vertex $v_i$ of degree 2 corresponding to $G_{S_i}$, otherwise by Claim 1 there is an $M(B)$-path of length a multiple of 3, which can be combined into a bigger $M$-3-block in $G$, a contradiction to the maximality of an $M$-3-block. Now there are two vertices $u, v$ in $S_i$ adjacent to $u', v'$ not in $G_{S_i}$ respectively.

Let $G'$ be the graph obtained from $G$ by contracting only the $M$-3-block $G_{S_i}$ into a vertex $v_i$. As $v_i$ is of degree 2 in $G^*$, we have $\sigma(G') \leq \sigma(G)$ and $G'$ is a 2-connected simple graph in which the length of every cycle is a multiple of 3. By the minimality of $G$, the graph $G'$ has a strong $\sigma(G)$-edge-coloring $f'$. Without loss of generality, we may assume that $f'(uu') \neq \sigma(G)$ and $f'(vv') \neq \sigma(G)$, as $\sigma(G) \geq 3$.

Similar to the proof of Claim 2, $G_{S_i} = S(I, 3)$ for some multi-graph $I$. Notice that $G_{S_i}$ and $H = S(I, 2)$ have the same maximum degree $\Delta$ and also $\sigma(G) \geq \sigma(G_{S_i}) = \Delta + 1 \geq 3$. Also, $H$ is a bipartite graph with one partite set $X = V(I)$ and the other partite set $Y = V(H) - X$ in which each vertex is of degree 2. Consider the list $L(u) \subseteq \{1, 2, \ldots, \sigma(G) - 1\} - \{f'(x'z): z \in N(x')\}$ with $|L(u)| = d_H(u)$, $L(v) \subseteq \{1, 2, \ldots, \sigma(G) - 1\} - \{f'(y'w): w \in N(y')\}$ with $|L(v)| = d_H(v)$, and $L(x) = \{1, 2, \ldots, d_H(x)\}$ for all other $x \in X$. By Lemma 15, $H$ has a proper edge-coloring $g$ using colors in $\{1, 2, \ldots, \sigma(G) - 1\}$. We may consider $G_{S_i} = S(I, 3)$ as a graph obtained from $H = S(I, 2)$ by inserting an edge between two edges incident to a vertex of degree two in $H$. Then we may extend the edge-coloring $g$ of $H$ to a strong edge-coloring $g'$ of $G_{S_i}$ by coloring all such newly inserted edges by $\sigma(G)$. This together with $f'$ give a strong $\sigma(G)$-edge-coloring. Hence, $\chi'_{s}(G) \leq \sigma(G)$ and so $\chi'_{s}(G) = \sigma(G)$, a contradiction to the assumption that $\chi'_{s}(G) > \sigma(G)$. \qed

Notice that the condition 2-connectedness is essential in the theorem above. We even cannot replace it by 2-edge-connected, as shown in the following example. Let $G$ is the graph obtained from $C(3, 2)$ by joining each pair of degree-one vertices adjacent to each vertex in the center $C_3$. It is easy to see that $\sigma(G) = 7 < 9 = \sigma^*(G) = \text{am}(G) = \chi'_{s}(G)$.

We close this paper by a conjecture that 2-connected graphs in which the length of every cycle is a multiple of 3 are vertex $\sigma$-perfect.
References


