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Abstract

A strong edge-coloring of a graph is a function that assigns to each edge a color such that two edges within distance two apart receive different colors. The *strong chromatic index* of a graph is the minimum number of colors used in a strong edge-coloring. This paper determines strong chromatic indices of cacti, which are graphs whose blocks are cycles or complete graphs of two vertices. The proof is by means of jellyfish graphs.

1 Introduction

The coloring problem considered in this article has restrictions on edges within distance two apart. The *distance* between two edges e and e' in a graph is the minimum k for which there is a sequence e_1, e_2, \dots, e_k of distinct edges such that $e = e_1$, $e' = e_k$, and e_{i-1} shares an end vertex with e_i for $2 \leq i \leq k$. A *strong*

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edge-coloring of a graph is a function that assigns to each edge a color such that any two edges within distance two apart receive different colors. A *color class* of a strong edge-coloring is the set of all edges using the same color. A *strong k -edge-coloring* is a strong edge-coloring using at most k colors. An *induced matching* is an edge set in which two distinct edges are of distance at least two. Finding a strong k -edge-coloring is equivalent to partitioning the edge set of the graph into k induced matchings. The *strong chromatic index* of a graph G , denoted by $\chi'_s(G)$, is the minimum k such that G admits a strong k -edge-coloring.

Strong edge-coloring was first studied by Fouquet and Jolivet [11, 12] for cubic planar graphs. By a greedy algorithm, it is easy to see that $\chi'_s(G) \leq 2\Delta^2 - 2\Delta + 1$ for any graph G of maximum degree Δ . Fouquet and Jolivet [11] established a Brooks type upper bound $\chi'_s(G) \leq 2\Delta^2 - 2\Delta$, which is not true only for $G = C_5$ as pointed out by Shiu and Tam [26]. The following conjecture was posed by Erdős and Nešetřil [8, 9] and revised by Faudree, Gyárfás, Schelp and Tuza [10]:

Conjecture 1. *If G is a graph of maximum degree Δ , then $\chi'_s(G) \leq \Delta^2 + \lfloor \frac{\Delta}{2} \rfloor^2$.*

For graphs with maximum degree $\Delta = 3$, Conjecture 1 was verified by Andersen [1] and by Horák, Qing and Trotter [15] independently. For $\Delta = 4$, while Conjecture 1 says that $\chi'_s(G) \leq 20$, Horák [14] obtained $\chi'_s(G) \leq 23$ and Cranston [7] proved $\chi'_s(G) \leq 22$. Molloy and Reed [22] proved that for large Δ every graph of maximum degree Δ has $\chi'_s(G) \leq 1.998\Delta^2$ using probabilistic method. Mahdian [19] proved that for a C_4 -free graph G , $\chi'_s(G) \leq (2 + o(1))\Delta^2 / \ln \Delta$. Faudree, Gyárfás, Schelp and Tuza [10] proved that for graphs where all cycle lengths are multiples of four, $\chi'_s(G) \leq \Delta^2$. They mentioned that this result could probably be improved to a linear function of the maximum degree. Brualdi and Massey [2] improved the upper bound to $\chi'_s(G) \leq \alpha\beta$ for such graphs, where α and β are the maximum degrees of the respective partitions. Nakprasit [23] proved that if G is bipartite and the maximum degree of one partite set is at most 2, then $\chi'_s(G) \leq 2\Delta$. Chang and Narayanan [6] proved that $\chi'_s(G) \leq 8\Delta - 6$ for chordless graphs G . This settles the above question by Faudree, Gyárfás, Schelp and Tuza [10] in the positive, since graphs with cycle lengths divisible by 4 are chordless graphs. They also established that $\chi'_s(G) \leq 10\Delta - 10$ for 2-degenerate graphs G .

Strong edge-coloring on planar graphs is also extensively studied in the literature. Faudree, Gyárfás, Schelp and Tuza [10] asked whether $\chi'_s(G) \leq 9$ if G is cubic planar. If this upper bound is proved to be true, it would be the best possible.

Faudree, Gyárfás, Schelp and Tuza [10] used the Four-color theorem to show that $\chi'_s(G) \leq 4\Delta(G) + 4$ for any planar graph G of maximum degree Δ . They also exhibited a planar graph G whose strong chromatic index is $4\Delta(G) - 4$. Their proof also gives a consequence that $\chi'_s(G) \leq 3\Delta$ for planar graphs G of girth at least 7. Chang, Montassier, Pecher and Raspaud [5] further proved that $\chi'_s(G) \leq 2\Delta - 1$ for planar graphs G with large girth. Strong chromatic index for Halin graphs was first considered by Shiu, Lam and Tam [25] and then studied in [4, 16, 18, 26]. For trees G they obtained that $\chi'_s(G) = \sigma(G)$, where

$$\sigma(G) := \max_{uv \in E(G)} (d_G(u) + d_G(v) - 1) \quad (1)$$

is an easy lower bound of $\chi'_s(G)$, that is,

$$\sigma(G) \leq \chi'_s(G) \text{ for any graph } G. \quad (2)$$

An edge xy in a graph G is σ -tight if $d_G(x) + d_G(y) - 1 = \sigma(G)$. Liao [17] studied cacti, which are graphs whose blocks are cycles or complete graphs of two vertices. Notice that cacti are planar graphs and include trees. He established that for a cactus G , $\chi'_s(G) = \sigma(G)$ if the length of any cycle is a multiple of 6, $\chi'_s(G) \leq \sigma(G) + 1$ if the length of any cycle is even, and $\chi'_s(G) \leq \lfloor \frac{3\sigma(G)+1}{2} \rfloor$ in general. For other results on strong edge-coloring, see [3, 13, 20, 21, 24, 27].

The purpose of this paper is to determine strong chromatic indices of cacti. The method is by means of jellyfish graphs to be introduced later. We first establish a decomposition theorem saying that the strong chromatic index of a graph is the maximum strong chromatic index of a block-jellyfish, which is a block together with edges with one vertex in the block and the other outside. Then we determine the strong chromatic index of a C_n -jellyfish, which is a graph obtained from the cycle C_n by attaching pendent edges to the cycle vertices.

2 Preliminary

For an integer $n \geq 3$, the n -cycle is the graph C_n with vertex set $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n\}$, where $v_{n+1} = v_1$. More generally, when the indices of the vertices of an n -cycle are arithmetic expressions, they are considered to be taken modulo n .

A *cut-vertex* of a graph is a vertex whose removing results in a graph with more components than the old graph. A *block* of a graph is a maximal connected subgraph

without cut-vertices in itself. Any two blocks of a graph have at most one vertex in common, and if they meet at one vertex, then it is a cut-vertex. An *end block* is a block with exactly one cut-vertex. A *block graph* is a graph whose blocks are complete graphs. A *cactus* is a graph whose blocks are cycles or complete graphs of two vertices.

For a graph H , the H -jellyfish $H(p_v: v \in V(H))$ is the graph obtained from H by adding p_v new vertices adjacent to v for each vertex v in H . An edge joining a new vertex to v is called a *pendent edge* at v . For a block H of a graph G , any vertex $u \in V(G) - V(H)$ is adjacent to at most one vertex $v \in V(H)$, and if the vertex v exists then it is a cut-vertex of G . A *block-jellyfish* of a graph G is the H -jellyfish H' for some block H of G , where the new vertices of H' are all vertices of $V(G) - V(H)$ having exactly one neighbor in $V(H)$. A block-jellyfish is *trivial* if it is an H -jellyfish for an end block H which is K_2 , otherwise it is *non-trivial*.

Lemma 2. *If H is a subgraph of G , then $\chi'_s(H) \leq \chi'_s(G)$.*

As any three consecutive edges in C_n use different colors in a strong edge-coloring, the following lemma is an easy consequence of parity checking.

Proposition 3. *If $n \geq 3$, then $\chi'_s(C_n) = 5$ for $n = 5$, $\chi'_s(C_n) = 3$ for n is a multiple of 3 and $\chi'_s(C_n) = 4$ otherwise.*

Notice that a trivial block-jellyfish H'_1 is a star; and if it is not a component, then it is a subgraph of a non-trivial block-jellyfish H'_2 . By Lemma 2, $\chi'_s(H'_1) \leq \chi'_s(H'_2)$.

Theorem 4. *Suppose G is a connected graph that is not a star. If G has exactly r non-trivial block-jellyfishes G_1, G_2, \dots, G_r , then $\chi'_s(G) = \max_{1 \leq i \leq r} \chi'_s(G_i)$.*

Proof. Since the graphs G_i are subgraphs of G , by Lemma 2, $\chi'_s(G) \geq \max_{1 \leq i \leq r} \chi'_s(G_i)$. Next, we shall prove by induction on r that $\chi'_s(G) \leq \max_{1 \leq i \leq r} \chi'_s(G_i)$. In the case where $r = 1$, $G = G_1$ and so the inequality is clear. Assume $r \geq 2$. Suppose the corresponding block of G_i in G is H_i . Then there is some H_i , say H_1 , which meets exactly one H_j at a cut-vertex of G . Let G' be obtained from G by deleting G_1 but keeping those vertices and edges in G_j undeleted. Then the non-trivial block-jellyfishes of G' are exactly G_2, G_3, \dots, G_r . By the induction hypothesis, $\chi'_s(G') \leq \max_{2 \leq i \leq r} \chi'_s(G_i)$. Color G' with $\chi'_s(G')$ colors. Since every two edges in $E(G_1) \cap E(G_j)$ are adjacent, meeting at the cut-vertex, we may assume that edges in $E(G_1) \cap E(G_j)$ are

colored by $\{1, 2, \dots, |E(G_1) \cap E(G_j)|\}$. On the other hand, since $d(e, e') > 2$ for any $e \in E(G_1) - E(G_j)$ and $e' \in E(G) - E(G_1)$, we can color edges in $E(G_1)$ by $\{1, 2, \dots, \chi'_s(G_1)\}$. Hence, we have colored G by $\max\{\chi'_s(G'), \chi'_s(G_1)\} \leq \max_{1 \leq i \leq r} \chi'_s(G_i)$ colors. \square

As an easy consequence, we have the following results for block graphs.

Corollary 5. *If G is a block graph, then $\chi'_s(G) = \max\{|E(H)|: H \text{ is a non-trivial block-jellyfish of } G\}$.*

Proof. This follows from Theorem 4 and the fact that any two edges in H are of distance within two. \square

Corollary 6. *If G is a C_3 -jellyfish, then $\chi'_s(G) = |E(G)|$.*

Proof. This follows from Corollary 5 and the fact that G is the only non-trivial block-jellyfish of itself. \square

Lemma 7. *If $G = H(p_v: v \in V(H))$ is an H -jellyfish such that $\{v: p_v \neq 0\} \subseteq X \cup Y$ for two independent sets X and Y , then $\chi'_s(G) \leq \chi'_s(H) + \max\{p_u + p_v: u \in X, v \in Y, uv \in E(H)\}$.*

Proof. Let $s = \max\{p_u + p_v: u \in X, v \in Y, uv \in E(H)\}$. For each vertex $u \in X$, color the pendent edges incident to u by $\{1, 2, \dots, p_u\}$, and for each vertex $v \in Y$, color the pendent edges incident to v by $\{s - p_v + 1, s - p_v + 2, \dots, s\}$. We verify that the coloring is legal. In fact, if a pendent edge uu' is within distance two from a pendent edge vv' , then $uv \in E(H)$. The assumption $p_u + p_v \leq s$ gives $p_u < s - p_v + 1$, so uu' and vv' are colored differently. We then use $s + 1, s + 2, \dots, s + \chi'_s(H)$ to color the edges of H . These give a strong edge-coloring of G and the lemma follows. \square

Corollary 8. *If G is a C_n -jellyfish with even n , then $\chi'_s(G) \leq \sigma(G) + \chi'_s(C_n) - 3$.*

Proof. Let $X = \{v_i: i \text{ is odd}\}$ and $Y = \{v_i: i \text{ is even}\}$. The corollary follows from Lemma 7 and the fact that $\max_{1 \leq i \leq n} (p_i + p_{i+1}) = \sigma(G) - 3$. \square

Corollary 9. ([17]) *If G is a C_n -jellyfish with even n , then $\chi'_s(G) \leq \sigma(G) + 1$.*

Proof. This follows from Corollary 8 and the fact that $\chi'_s(C_n) \leq 4$. \square

Corollary 10. *If G is a C_4 -jellyfish, then $\chi'_s(G) = \sigma(G) + 1$.*

Proof. By Corollary 9, $\chi'_s(G) \leq \sigma(G) + 1$. On the other hand, consider a cycle edge xy such that $d_G(x) + d_G(y) - 1 = \sigma(G)$. Since the cycle edge not incident to x or y is within distance 2 from the edges incident to x or y , we have $\chi'_s(G) \geq \sigma(G) + 1$, so $\chi'_s(G) = \sigma(G) + 1$. \square

Corollary 11. ([17]) *If G is a C_n -jellyfish with n a multiple of 6, then $\chi'_s(G) = \sigma(G)$.*

Proof. This follows from Corollary 8 and the fact that $\chi'_s(C_n) = 3$. \square

Corollary 12. *Suppose G is a C_n -jellyfish with $d_G(v_j) = 2$ for some j . If $G \neq C_5$, then $\chi'_s(G) \leq \sigma(G) + 1$. If n is a multiple of 3, then $\chi'_s(G) = \sigma(G)$.*

Proof. Without loss of generality, we may assume that $j = n$. Let $X = \{v_i : i \neq n \text{ and } i \text{ is odd}\}$ and $Y = \{v_i : i \neq n \text{ and } i \text{ is even}\}$.

Since $\max_{1 \leq i \leq n-1} (p_i + p_{i+1}) = \sigma(G) - 3$, by Lemma 7, $\chi'_s(G) \leq \chi'_s(C_n) + \sigma(G) - 3$. In the case where $n \neq 5$, this and $\chi'_s(C_n) \leq 4$ together imply that $\chi'_s(G) \leq \sigma(G) + 1$. In the case where n is a multiple of 3, this and $\chi'_s(C_n) = 3$ together imply that $\chi'_s(G) \leq \sigma(G)$, so $\chi'_s(G) = \sigma(G)$.

For the case of $n = 5$, consider the C_5 -jellyfish $H = C_5(\min\{p_i, 1\} : 1 \leq i \leq 5)$. Notice that every cycle vertex of H has at most one pendent edge. Then $\chi'_s(H) \leq 5$, since we can color the edges of H with 5 colors by coloring the pendent edge at v_i (if any) with the same color as the cycle edge $v_{i+2}v_{i+3}$, where the indices are taken modulo 5. Let $p'_i = p_i - \min\{p_i, 1\}$ for $1 \leq i \leq 5$. Notice that $p_5 = p'_5 = 0$ and $\max_{1 \leq i \leq 4} (p'_i + p'_{i+1}) \leq \sigma(G) - 4$, since there is at least one $p_i \neq 0$. Then G is the H -jellyfish $H(p'_i : 1 \leq i \leq 5)$, where the un-presented $p_u = 0$ for all leaves u of H . By Lemma 7, $\chi'_s(G) \leq \max_{1 \leq i \leq 4} (p'_i + p'_{i+1}) + \chi'_s(H) \leq \sigma(G) - 4 + 5 = \sigma(G) + 1$. \square

3 Strong edge-coloring on cacti

The purpose of this section is to give the strong chromatic indices of cacti. Notice that a block-jellyfish of a cactus is either a K_2 -jellyfish or a C_n -jellyfish. The strong chromatic index of a K_2 -jellyfish is equal to its number of edges. So we only need to consider the case of C_n -jellyfish. Now suppose that G is a C_n -jellyfish. Notice that $G = C_n(p_1, p_2, \dots, p_n)$, where $p_i = d_G(v_i) - 2$ for $1 \leq i \leq n$. A rotation of a C_n -jellyfish $G = C_n(p_1, p_2, \dots, p_n)$ is a C_n -jellyfish $G' = C_n(p'_1, p'_2, \dots, p'_n)$ with all

$p'_i = p_{i+r}$ for a constant r , where the index in p_{i+r} is taken modulo n . As we have the values for cycles in Proposition 3, we may only consider the case of $\sigma(G) \geq 4$.

Theorem 13. *If G is a C_n -jellyfish of m edges with $\sigma(G) \geq 4$, then $\chi'_s(G) =$*

$$\left\{ \begin{array}{l} m, \quad \text{if } n = 3; \\ \sigma(G) + 1, \quad \text{if } n = 4; \\ \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil, \quad \text{otherwise, if } n \text{ is odd with all } d_G(v_i) = d \text{ but } (n, d) \neq (7, 3), \\ \quad \text{or } \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil \geq \sigma(G) + 1; \\ \sigma(G) + 1, \quad \text{otherwise, if } (n, d) = (7, 3) \text{ with all } d_G(v_i) = d, \\ \quad \text{or } n \not\equiv 0 \pmod{3} \text{ such that up to rotation} \\ \quad \quad d_G(v_i) = \sigma(G) - 1 \text{ for } i \equiv 1 \pmod{3} \text{ with } 1 \leq i \leq 3 \lfloor \frac{n}{3} \rfloor - 2, \\ \quad \quad \text{or } (n, \sigma(G)) = (10, 4) \text{ with } d_G(v_i) = 3 \text{ for all odd or all even } i; \\ \sigma(G), \quad \text{otherwise.} \end{array} \right.$$

To prove the main theorem, we first establish a sequence of lemmas as follows.

Lemma 14. *If G is a C_n -jellyfish graph of m edges, then any color class of a strong edge-coloring has at most $\lfloor n/2 \rfloor$ edges and $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil \leq \chi'_s(G)$.*

Proof. We claim that there are at most $\lfloor n/2 \rfloor$ edges using the same color in a strong edge-coloring of G . For $1 \leq i \leq n$, consider the set E_i consisting all edges incident to v_i or v_{i+1} except the edge $v_{i-1}v_i$. Then for a fixed color c , each E_i contains at most one edge colored by c . As each edge of G appears in exactly two sets in E_1, E_2, \dots, E_n , there are at most $\lfloor n/2 \rfloor$ edges using the color c . Hence $\frac{m}{\lfloor n/2 \rfloor} \leq \chi'_s(G)$ and so $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil \leq \chi'_s(G)$. \square

Lemma 15. *If n is even or $d_G(v_j) = 2$ for some j , then $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil \leq \sigma(G)$. If n is odd and $d_G(v_i) = d$ for $1 \leq i \leq n$, then $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil = \sigma(G)$ for $2 \leq d \leq (n+1)/2$, $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil = \sigma(G) + 1$ for $(n+3)/2 \leq d \leq n$ and $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil \geq \sigma(G) + 2$ for $d \geq n+1$.*

Proof. If n is even or $d_G(v_j) = 2$ for some j , say $j = n$, then $m-1 \leq \sum_{i=1}^{\lfloor n/2 \rfloor} (d_G(v_{2i-1}) + d_G(v_{2i}) - 2) \leq \lfloor n/2 \rfloor (\sigma(G) - 1)$, so $\frac{m}{\lfloor n/2 \rfloor} \leq \sigma(G)$. If n is odd and $d_G(v_i) = d$ for $1 \leq i \leq n$, then $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil = 2d - 2 + \lceil \frac{2d-2}{n-1} \rceil = \sigma(G) - 1 + \lceil \frac{2d-2}{n-1} \rceil$, which is $\sigma(G)$ for $2 \leq d \leq (n+1)/2$, is $\sigma(G) + 1$ for $(n+3)/2 \leq d \leq n$ and is at least $\sigma(G) + 2$ for $d \geq n+1$. \square

Lemma 16. *If G is a C_n -jellyfish with $d_G(v_i) = d \geq 3$ for $1 \leq i \leq n$, then*

$$\chi'_s(G) = \begin{cases} \sigma(G) + 1, & \text{if } n = 4 \text{ or } (n, d) = (7, 3); \\ \sigma(G), & \text{if } n \geq 6 \text{ is even}; \\ \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil, & \text{if } n \geq 3 \text{ is odd but } (n, d) \neq (7, 3). \end{cases}$$

Proof. **Case 1.** $n = 4$. In this case, the lemma follows from Corollary 10.

Case 2. $(n, d) = (7, 3)$. In this case, an induced matching has at most 3 edges by Lemma 14. If $\chi'_s(G) \leq 5$, then there are at least two color classes containing 2 cycle edges, each of which must be precisely of size 2. Then all color classes contain at most 13 edges, contradicting to the fact that G has 14 edges. Hence $\chi'_s(G) \geq 6$. Figure 1 gives a strong 6-edge-coloring of G , so $\chi'_s(G) = 6 = \sigma(G) + 1$.

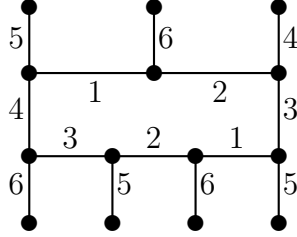


Figure 1: A strong 6-edge-coloring of G .

Case 3. $n \geq 6$ is even. In this case, by Corollary 11, we only need to consider the case of $n \not\equiv 0 \pmod{3}$. For $1 \leq i \leq n$, let $e_i = v_i v_{i+1}$ and $f_{i,1}, f_{i,2}, \dots, f_{i,d-2}$ be the pendent edges at v_i . The lemma follows from the fact that we may partition $E(G)$ into $\sigma(G) = 2d - 1$ induced matchings as follows:

$$\begin{aligned} & \text{for } n \equiv 2 \pmod{3}, \begin{cases} M_1 = \{f_{1,1}, e_3\} \cup \{e_i : 6 \leq i \leq n, i \equiv 0 \pmod{3}\}, \\ M_2 = \{f_{3,1}, e_5\} \cup \{e_i : 6 \leq i \leq n, i \equiv 2 \pmod{3}\}, \\ M_3 = \{f_{5,1}, e_2\} \cup \{e_i : 6 \leq i \leq n, i \equiv 1 \pmod{3}\}, \\ M_4 = \{e_1, e_4\} \cup \{f_{i,1} : 7 \leq i \leq n, i \equiv 1 \pmod{2}\}, \\ M_5 = \{f_{i,1} : 2 \leq i \leq n, i \equiv 0 \pmod{2}\}; \end{cases} \\ & \text{for } n \equiv 1 \pmod{3}, \begin{cases} M_1 = \{f_{1,1}, f_{6,1}, e_3\} \cup \{e_i : 6 \leq i \leq n, i \equiv 2 \pmod{3}\}, \\ M_2 = \{f_{2,1}, f_{4,1}\} \cup \{e_i : 6 \leq i \leq n, i \equiv 0 \pmod{3}\}, \\ M_3 = \{f_{3,1}, f_{5,1}\} \cup \{e_i : 6 \leq i \leq n, i \equiv 1 \pmod{3}\}, \\ M_4 = \{e_1, e_4\} \cup \{f_{i,1} : 7 \leq i \leq n, i \equiv 1 \pmod{2}\}, \\ M_5 = \{e_2, e_5\} \cup \{f_{i,1} : 7 \leq i \leq n, i \equiv 0 \pmod{2}\}; \end{cases} \\ & \text{for } 2 \leq j \leq d-2, \begin{cases} M_{2j+2} = \{f_{i,j} : 1 \leq i \leq n, i \equiv 1 \pmod{2}\}, \\ M_{2j+3} = \{f_{i,j} : 1 \leq i \leq n, i \equiv 0 \pmod{2}\}. \end{cases} \end{aligned}$$

Case 4. $n \geq 3$ is odd but $(n, d) \neq (7, 3)$. In this case, $\lfloor n/2 \rfloor = (n-1)/2$, $m = n(d-1)$ and $\frac{m}{\lfloor n/2 \rfloor} = 2n(d-1)/(n-1)$. By Lemma 14, $\chi'_s(G) \geq \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil$. We shall show the upper bound by considering two subcases.

Case 4-1. $3 \leq d \leq (n+1)/2$. We will show that $\chi'_s(G) \leq 2d-1$ and since $2d-1 < \frac{m}{\lfloor n/2 \rfloor} + 1$, $\chi'_s(G) = \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil$. For any integer t and odd q with $1 \leq q \leq n/3$, let

$$I(t, q) = \{e_{t+3}, e_{t+6}, e_{t+9}, \dots, e_{t+3q}\} \cup \{f_{t+3q+3}, f_{t+3q+5}, f_{t+3q+7}, \dots, f_{t+n+1}\},$$

where the indices are taken modulo n and we denote every pendent edge at v_i by f_i for simplicity. This is an induced matching containing q cycle edges and $(n-3q)/2$ pendent edges. Since $5 \leq 2d-1 \leq n$, n is odd and $(n, d) \neq (7, 3)$, we can write n as a sum of $2d-1$ odd numbers $q_1, q_2, \dots, q_{2d-1}$, each of which is no more than $n/3$. This can be done by choosing q_i 's such that the gap between the maximum and the minimum is at most 2. Let $Q_i = \sum_{j=1}^i 3q_j$ for $0 \leq i \leq 2d-1$.

In the case where n is not a multiple of 3, we claim that the induced matchings $I(Q_{i-1}, q_i)$, for $1 \leq i \leq 2d-1$, partition $E(G)$. First, the cycle edges used are $e_3, e_6, e_9, \dots, e_{3q_1}; e_{3q_1+3}, e_{3q_1+6}, e_{3q_1+9}, \dots, e_{3q_1+3q_2}; \dots; e_{Q_{2d-2}+3}, e_{Q_{2d-2}+6}, e_{Q_{2d-2}+9}, \dots, e_{Q_{2d-2}+3q_{2d-1}} = e_{3n}$, which cover each cycle edge exactly once as n is not a multiple of 3. Secondly, the pendent edges used, viewing backward, are $f_{n+1}, f_{n-1}, f_{n-3}, \dots, f_{3q_1+3}; f_{3q_1+1}, f_{3q_1-1}, f_{3q_1-3}, \dots, f_{3q_1+3q_2+3}; \dots; f_{Q_{2d-2}+1}, f_{Q_{2d-2}-1}, f_{Q_{2d-2}-3}, \dots, f_{Q_{2d-2}+3q_{2d-1}+3} = f_{n+3}$, which cover pendent edges at each cycle vertex exactly $\frac{1}{n} \sum_{i=1}^{2d-1} (n-3q_i)/2 = d-2$ times.

In the case where n is a multiple of 3, we modify the above arguments as follows. We may assume that $n \geq 9$ as the case for $n=3$ follows from Corollary 6. In this case, we may choose $q_1, q_2, \dots, q_{2d-1}$ so that they can be divided into 3 parts, each summing up to $n/3$. This can be done by first choosing q_1, q_2, q_3 to be $n/3, n/3, n/3$, and then properly shifting 1's from them to other q_i 's. We adopt similar arguments as above for the 3 parts separately, but consider rather $Q_i = r + \sum_{j=1}^i 3q_j$ for the r th part, $1 \leq r \leq 3$. The induced matchings in part r cover all the $n/3$ cycle edges e_i with $i \equiv r \pmod{3}$, and the pendant edges $f_{r+n+1}, f_{r+n-1}, f_{r+n-3}, \dots, f_{r+3(3n)+3}$, with a total number of a multiple of n . Similarly, pendant edges at each cycle vertex are also covered $d-2$ times.

Case 4-2. $d > (n+1)/2$. In this case, we partition the edges of G into two parts: the first part consists of the cycle edges together with $(n-3)/2$ pendent edges at each cycle vertex, and the second part consists of $d - (n+1)/2$ pendent

edges at each cycle vertex. The first part has $m_1 = n(n-1)/2$ edges. By Case 4-1, it can be partitioned into n induced matchings. Next, we order the pendant edges in the second part as $h_1, h_2, \dots, h_{m-m_1}$, where h_j is a pendant edge at cycle vertex v_i with $i \equiv 2j - 1 \pmod{n}$. Notice that for any integer t and any integer $r \leq (n-1)/2$, the set $\{h_{t+1}, h_{t+2}, \dots, h_{t+r}\}$ is an induced matching. Hence the second part can be partitioned into $\lceil \frac{m-m_1}{(n-1)/2} \rceil$ induced matchings. Totally, the edges of G can be partitioned into $\lceil \frac{m}{(n-1)/2} \rceil$ as desired. \square

We now consider the case where $d_G(v_j) = 2$ for some v_j , say v_n . By Corollary 12, $\chi'_s(G) = \sigma(G)$ or $\sigma(G) + 1$.

Lemma 17. *If $n \not\equiv 0 \pmod{3}$ and G is a C_n -jellyfish such that $d_G(v_i) = \sigma(G) - 1$ for $i \equiv 1 \pmod{3}$ and $1 \leq i \leq 3\lfloor n/3 \rfloor - 2$, then $\chi'_s(G) = \sigma(G) + 1$.*

Proof. First, the assumption gives that $d_G(v_j) = 2$ for $j = n$ or $j \not\equiv 1 \pmod{3}$ with $1 \leq j \leq 3\lfloor n/3 \rfloor - 1$. By Corollary 12, $\chi'_s(G) \leq \sigma(G) + 1$. Suppose to the contrary that G had a strong edge-coloring using $\sigma(G)$ colors. Then for each $i \equiv 1 \pmod{3}$ with $1 \leq i \leq 3\lfloor n/3 \rfloor - 2$, the $\sigma(G) - 3$ pendent edges at v_i , e_{i-1} , e_i , together with e_{i-2} (respectively, e_{i+1}) would use all the $\sigma(G)$ colors. It follows that e_{i-2} and e_{i+1} would use the same color. Hence $e_{n-1}, e_2, e_5, \dots, e_{3\lfloor n/3 \rfloor - 1}$ would all use the same color. Since $n \not\equiv 0 \pmod{3}$, e_{n-1} and $e_{3\lfloor n/3 \rfloor - 1}$ are two distinct edges within distance two, which leads to a contradiction. \square

Lemma 18. *If G is a C_{10} -jellyfish such that $d_G(v_i) = \sigma(G) - 1 = 3$ for all odd i , then $\chi'_s(G) = \sigma(G) + 1 = 5$.*

Proof. First, the assumption gives that $d_G(v_j) = 2$ for all even j . By Corollary 12, $\chi'_s(G) \leq \sigma(G) + 1$. Suppose to the contrary that G had a strong edge-coloring using $\sigma(G) = 4$ colors. Then for each odd i , the $\sigma(G) - 3 = 1$ pendent edge at v_i , e_{i-1} , e_i , together with e_{i-2} (respectively, e_{i+1}) would use all the $\sigma(G)$ colors. This gives that e_{i-2} and e_{i+1} use the same color. Since we only had 4 colors for the 10 cycle edges, there would be one color used for at least 3 edges. But a color should appear in a pair of edges as shown above. This color would then be used for at least 4 cycle edges, which is impossible. \square

Lemma 19. *If G is a C_n -jellyfish with $\sigma(G) = 4$, then $\chi'_s(G) = \sigma(G)$ except that $\chi'_s(G) = \sigma(G) + 1$ when, up to rotation, $n \not\equiv 0 \pmod{3}$ such that $d_G(v_i) = 3$ for $i \equiv 1 \pmod{3}$ with $1 \leq i \leq 3\lfloor \frac{n}{3} \rfloor - 2$ or $n = 10$ such that $d_G(v_i) = 3$ for all odd i .*

Proof. If n is a multiple of 3, then the lemma follows from Corollary 12. Now assume that $n \not\equiv 0 \pmod{3}$. The exceptional cases follow from Lemmas 17 and 18.

Up to rotation, we may assume that $1 = i_1 < i_2 < \dots < i_s$ are all the indices for which v_{i_r} is of degree 3. For $1 \leq r \leq s$, the path P_r from v_{i_r} to $v_{i_{r+1}}$ consists of $n_r = i_{r+1} - i_r$ cycle edges, where $i_{s+1} = n + 1$. Using this notion, the C_n -jellyfish G is completely determined by the sequence n_1, n_2, \dots, n_s . Notice that the first exceptional case is the same as that all $n_r = 3$ except exactly one $n_r \in \{2, 4, 5\}$ or exactly two consecutive $n_r = 2$, and the second exceptional case is the same as that $n = 10$ and all $n_r = 2$. We consider cases other than the two exceptional cases. Since the cases for $n = 4, 5$ are included in the first exceptional case, and 6 is a multiple of 3, we may assume $n \geq 7$. The aim is to find a strong 4-edge-coloring for G . By adding suitable pendent edges, we may assume that all $n_r \in \{2, 3\}$ and there are two non-consecutive $n_r = 2$.

If there is at least one $n_r = 3$, then up to rotation we may assume that $n_s = 2$, and there exists some $t \leq s - 1$ such that $n_r = 3$ for all $1 \leq r \leq t$ and $n_{t+1} = 2$. Otherwise, if all $n_r = 2$, then $s \geq 4$, $s \neq 5$, and we choose $t = 0$. We define an edge-coloring on cycle edges first as follows.

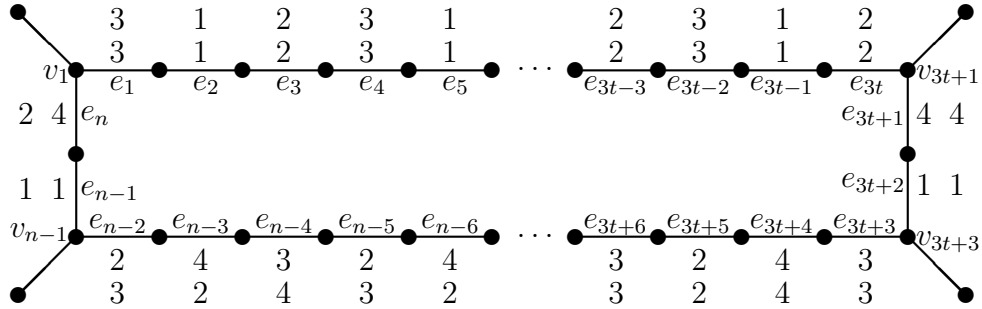


Figure 2: Inner labels are for $n \equiv 1 \pmod{3}$ and outer labels are for $n \equiv 2 \pmod{3}$.

Notice that in the coloring as shown in Figure 2, the edges in the upper path $e_1, e_2, e_3, \dots, e_{3t-2}, e_{3t-1}, e_{3t}$ are colored by 3, 1, 2 periodically; the edges in the lower path $e_{3t+3}, e_{3t+4}, e_{3t+5}, \dots, e_{n-3-x}, e_{n-2-x}, e_{n-1-x}$ are colored by 3, 4, 2 periodically, where $x = (n \bmod 3)$. These colors for the cycle edges satisfy the following two conditions.

- (i) Any two distinct cycle edges within distance two receive distinct colors.
- (ii) The two cycle edges with distance exactly two from a pendent edge receive a same color.

By (ii), the four cycle edges within distance two from a pendent edges use only 3 colors. Hence we may color any pendent edge with the remaining color to form a strong 4-edge-coloring of G . \square

Lemma 20. *If G is a C_n -jellyfish with $\sigma(G) \geq 4$ and $d_G(v_j) = 2$ for some j , then $\chi'_s(G) = \sigma(G)$ except that $\chi'_s(G) = \sigma(G) + 1$ when, up to rotation, $n \not\equiv 0 \pmod{3}$ such that $d_G(v_i) = \sigma(G) - 1$ for $i \equiv 1 \pmod{3}$ with $1 \leq i \leq 3\lfloor \frac{n}{3} \rfloor - 2$ or $n = 10$ such that $d_G(v_i) = \sigma(G) - 1 = 3$ for all odd i .*

Proof. The exceptional cases follow from Lemmas 17 and 18. We shall prove the lemma by induction on $\sigma(G)$. The case of $\sigma(G) = 4$ follows from Lemma 19. Now assume that $\sigma(G) \geq 5$.

A *run* is a maximal sequence $v_i, v_{i+1}, \dots, v_{i+j}$ of cycle vertices in which every vertex is of degree at least 3. The *even-half* (respectively, *odd-half*) of the run is the vertices v_{i+r} with $0 \leq r \leq j$ and r even (respectively, odd). Notice that an even-half of a run is always non-empty, while an odd-half is empty if and only if $j = 0$. Consider a C_n -jellyfish G' obtained from G by deleting a pendent edge at each vertex of exactly one of the even-half or the odd-half of each run. Then $\sigma(G) = \sigma(G') + 1$ and $\chi'_s(G) \leq \chi'_s(G') + 1$ as the deleted edges form an induced matching.

Suppose that G' is not in the exceptional cases. By the induction hypothesis, $\chi'_s(G') = \sigma(G')$. Then $\chi'_s(G) \leq \chi'_s(G') + 1 = \sigma(G') + 1 = \sigma(G)$, so $\chi'_s(G) = \sigma(G)$. Now we may assume that G' is in the exceptional cases. If there is a run of length one in G' obtained from some run of length not one in G , then we change to delete the other half of this run in G and obtain a new G' which is not in the exceptional cases. Now every run of length one in G' is obtained from a run of length one in G , and since G' is in the exceptional cases but not G , it must be that $n = 10$ and $d_G(v_i) = \sigma(G) - 1 = 4$ for all odd i . Then $\chi'_s(G) = \sigma(G) = 5$ as shown in Figure 3. \square

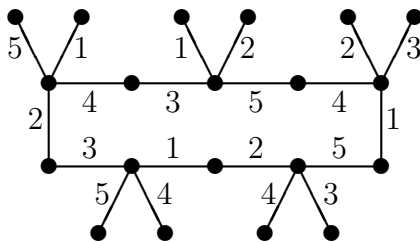


Figure 3: The C_{10} -jellyfish G with $d_G(v_i) = \sigma(G) - 1 = 4$ for all odd i .

Having the above lemmas established, we are now ready to prove Theorem 13. For the case of $n = 3$, the theorem follows from Corollary 6. For the case of $n = 4$, the theorem follows from Corollary 10. Now we may assume that $n \geq 5$.

If $d_G(v_j) = 2$ for some j , then $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil \leq \sigma(G)$ by Lemma 15, so the third case of the theorem does not happen. The theorem then follows from Lemma 20.

We now consider the case where $d_G(v_i) \geq 3$ for all i . There are two subcases to be considered depending on the parity of n .

We first consider the subcase where n is even. By Lemma 15, $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil \leq \sigma(G)$, so the third case of the theorem does not happen. We then only need to prove that $\chi'_s(G) = \sigma(G)$. Let H be the C_n -jellyfish with each cycle vertex v_i of degree 3. By Lemma 16, $\chi'_s(H) = 5$. Then G is obtained from H by adding $p_{v_i} = d_G(v_i) - 3$ pendent edges at v_i for $1 \leq i \leq n$. Let $X = \{v_i : 1 \leq i \leq n \text{ and } i \text{ odd}\}$ and $Y = \{v_i : 1 \leq i \leq n \text{ and } i \text{ even}\}$. Then $\max\{p_u + p_v : u \in X, v \in Y, uv \in E(H)\} = \sigma(G) - 5$. By Lemma 7, $\chi'_s(G) \leq \chi'_s(H) + \max\{p_u + p_v : u \in X, v \in Y, uv \in E(H)\} \leq 5 + \sigma(G) - 5 = \sigma(G)$ and so $\chi'_s(G) = \sigma(G)$.

Next we consider the second subcase when n is odd. If all cycle edges $v_i v_{i+1}$ are tight, then $d_G(v_i) + d_G(v_{i+1}) - 1 = d_G(v_{i+1}) + d_G(v_{i+2}) - 1$ and so $d_G(v_i) = d_G(v_{i+2})$ for all i . Since n is odd, all $d_G(v_i)$ are equal.

Suppose, up to rotation, that $v_n v_1$ is a non-tight edge. Consider the C_n -jellyfish graph G_1 obtained from $G_0 := G$ by deleting one pendent edge at v_i for all even i . Then $\sigma(G_1) = \sigma(G_0) - 1$ and G_1 has $m_1 = m - \lfloor n/2 \rfloor$ edges. Since we can use one color for the deleted edges, $\chi'_s(G_0) \leq \chi'_s(G_1) + 1$. Repeating the same process gives that there is an integer $s \geq 0$ and C_n -jellyfish graphs G_0, G_1, \dots, G_s such that $\sigma(G_r) \geq 4$, $\sigma(G_r) = \sigma(G) - r$, G_r has $m_r = m - r \lfloor n/2 \rfloor$ edges, $\chi'_s(G) \leq \chi'_s(G_r) + r$ for $0 \leq r \leq s$, and either $d_{G_s}(v_j) = 2$ for some j or else $d_{G_s}(v_i)$ is a constant d_s for all i .

For the former case, G_{s-1} has the property that all cycle vertices have degree at least 3. But after deleting $(n-1)/2$ pendent edges, the resulting graph G_s has some cycle vertex v_j with degree 2. It then must be the case that G_s is not in the exceptional cases in Lemma 20. Hence $\chi'_s(G_s) = \sigma(G_s)$ and $\chi'_s(G) \leq \chi'_s(G_s) + s = \sigma(G_s) + s = \sigma(G)$. By Lemma 15, $\frac{m_s}{\lfloor n/2 \rfloor} \leq \sigma(G_s)$, so $\frac{m}{\lfloor n/2 \rfloor} = \frac{m_s}{\lfloor n/2 \rfloor} + s \leq \sigma(G_s) + s = \sigma(G)$. It follows that G fits the fifth case.

Now we may assume that $d_{G_s}(v_i)$ is a constant d_s for all i . If $(n, d_s) \neq (7, 3)$, then

by Lemma 16, $\chi'_s(G_s) = \lceil \frac{m_s}{\lfloor n/2 \rfloor} \rceil$ and so $\chi'_s(G) \leq \chi'_s(G_s) + s = \lceil \frac{m_s}{\lfloor n/2 \rfloor} \rceil + s = \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil$. By Lemma 14, $\chi'_s(G) = \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil$. If $(n, d_s) = (7, 3)$, then G_{s-1} must be the graph as in Figure 4 from which we conclude that $\chi'_s(G_{s-1}) = \sigma(G_{s-1})$. Then $\chi'_s(G) \leq \chi'_s(G_{s-1}) + s - 1 = \sigma(G_{s-1}) + s - 1 = \sigma(G)$. Notice that $\lceil \frac{m_s}{\lfloor n/2 \rfloor} \rceil = 5 = \sigma(G_s)$ and $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil = \lceil \frac{m_s}{\lfloor n/2 \rfloor} \rceil + s = \sigma(G_s) + s = \sigma(G)$, so G fits the fifth case.

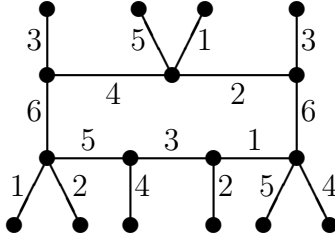


Figure 4: The graph G_{s-1} for $(n, d_s) = (7, 3)$.

References

- [1] L. D. Andersen, The strong chromatic index of a cubic graph is at most 10, *Discrete Math.* 108 (1992) 231-252.
- [2] R. A. Brualdi and J. J. Q. Massey, Incidence and strong edge colorings of graphs, *Discrete Math.* 122 (1993) 51-58.
- [3] K. Cameron, Induced matchings, *Discrete Appl. Math.* 24 (1989) 97-102.
- [4] G. J. Chang and D. D.-F. Liu, Strong edge-coloring for cubic halin graphs, *Discrete Math.* 312 (2012) 1468-1475.
- [5] G. J. Chang, M. Montassier, A. Pecher and A. Raspaud, Strong chromatic index of planar graphs with large girth, *Discuss. Math. Graph Theory* (accepted).
- [6] G. J. Chang and N. Narayanan, Strong chromatic index of 2-degenerate graphs, *J. Graph Theory* 73 (2013) 119-126.
- [7] D. Cranston, Strong edge-coloring graphs with maximum degree 4 using 22 colors, *Discrete Math.* 306 (2006) 2772-2778.
- [8] P. Erdős, Problems and results in combinatorial analysis and graph theory, *Discrete Math.* 72 (1988) 81-92.

- [9] P. Erdős and J. Nešetřil, [Problem], in: G. Halász and V. T. Sós (eds.), *Irregularities of Partitions*, Springer, Berlin, 1989, 162-163.
- [10] R. J. Faudree, A. Gyárfás, R. H. Schelp and Zs. Tuza, The strong chromatic index of graphs, *Ars Combin.* 29B (1990) 205-211.
- [11] J. L. Fouquet and J. Jolivet, Strong edge-coloring of graphs and applications to multi- k -gons, *Ars Combin.* 16A (1983) 141-150.
- [12] J. L. Fouquet and J. Jolivet, Strong edge-coloring of cubic planar graphs, *Progress in Graph Theory (Waterloo 1982)*, 1984, 247-264.
- [13] M. C. Golumbic and M. Lewenstein, New results on induced matchings, *Discrete Appl. Math.* 101 (2000) 157-165.
- [14] P. Horák, The strong chromatic index of graphs with maximum degree four, *Contemp. Methods Graph Theory*, 1990, 399-403.
- [15] P. Horák, H. Qing and W. T. Trotter, Induced matchings in cubic graphs, *J. Graph Theory* 17 (1993) 151-160.
- [16] H.-H. Lai, K.-W. Lih and P.-Y. Tsai, The strong chromatic index of Halin graphs, *Discrete Math.* 312 (2012) 1536-1541.
- [17] S.-T. Liao, The Strong Chromatic Index of Cacti, Master Thesis, Department of Mathematics, National Taiwan University, Taipei, Taiwan, June, 2012.
- [18] K.-W. Lih and D. D.-F. Liu, On the strong chromatic index of cubic Halin graphs, *Appl. Math. Lett.* 25 (2012) 898-901.
- [19] M. Mahdian, The strong chromatic index of C_4 -free graphs. *Random Stru. Algor.* 17 (2000) 357-375.
- [20] M. Mahdian, On the computational complexity of strong edge coloring, *Discrete Appl. Math.* 118 (2002) 239-248.
- [21] M. Maydanskiy, The incidence coloring conjecture for graphs of maximum degree 3, *Discrete Math.* 292 (2005) 131-141.
- [22] M. Molloy and B. Reed, A bound on the strong chromatic index of a graph, *J. Combin. Theory, Series B* 69 (1997) 103-109.

- [23] K. Nakprasit, A note on the strong chromatic index of bipartite graphs, *Discrete Math.* 308 (2008) 3726-3728.
- [24] M. R. Salavatipour, A polynomial time algorithm for strong edge coloring of partial k -trees, *Discrete Appl. Math.* 143 (2004) 285-291.
- [25] W. C. Shiu, P. C. B. Lam and W. K. Tam, On strong chromatic index of Halin graphs, *J. Combin. Math. Combin. Comput.* 57 (2006) 211-222.
- [26] W. C. Shiu and W. K. Tam, The strong chromatic index of complete cubic Halin graphs, *Appl. Math. Lett.* 22 (2009) 754-758.
- [27] J. Wu and W. Lin, The strong chromatic index of a class of graphs, *Discrete Math.* 308 (2008) 6254-6262.