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BPS States and the P = W Conjecture

W.-Y. Chuang, D.-E. Diaconescu, and G. PAN

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BPS STATES AND THE $P=W$ CONJECTURE

W.-Y. CHUANG[1], D.-E. DIACONESCU[2], G. PAN[2]

Abstract. A string theoretic framework is presented for the work of Hausel and Rodriguez-Vilegas as well as de Cataldo, Hausel and Migliorini on the cohomology of character varieties. The central element of this construction is an identification of the cohomology of the Hitchin moduli space with BPS states in a local Calabi-Yau threefold. This is a summary of several talks given during the Moduli Space Program 2011 at Isaac Newton Institute.

1. Introduction

Consider an M-theory compactification on a smooth projective Calabi-Yau threefold Y. M2-branes wrapping holomorphic curves in Y yield supersymmetric BPS states in the five dimensional effective action. These particles are electrically charged under the low energy $U(1)$ gauge fields. The lattice of electric charges is naturally identified with second homology lattice $H_2(Y,\mathbb{Z})$. Quantum states of massive particles in five dimensions also form multiplets of the little group $SU(2)_L\times SU(2)_R\subset Spin(4,1)$, which is the stabilizer of the time direction in $\mathbb{R}^5$. The unitary irreducible representations of $SU(2)_L\times SU(2)_R$ may be labelled by pairs of half-integers $(j_L,j_R)\in\left(\frac{1}{2}\mathbb{Z}\right)^2$. The half-integers $(j_L/2,j_R/2)$ are the left, respectively right moving spin quantum numbers. In conclusion, the space of five dimensional BPS states admits a direct sum decomposition

$$\mathcal{H}_{BPS}(Y)\simeq\bigoplus_{\beta\in H_2(Y,\mathbb{Z})}\ \bigoplus_{j_L,j_R\in\frac{1}{2}\mathbb{Z}}\mathcal{H}_{BPS}(Y,\beta,j_L,j_R).$$

The refined Gopakumar-Vafa invariants are the BPS degeneracies

$$N(Y,\beta,j_L,j_R)=\dim\mathcal{H}_{BPS}(Y,\beta,j_L,j_R).$$

The unrefined invariants are BPS indices,

$$N(Y,\beta,j_L)=\sum_{j_R\in\frac{1}{2}\mathbb{Z}}(-1)^{2j_R+1}(2j_R+1)N(Y,\beta,j_L,j_R).$$

String theory arguments [12] imply that BPS states should be identified with cohomology classes of moduli spaces of stable pure dimension sheaves on Y. More specifically, let $\mathcal{M}(Y,\beta,n)$ be the moduli space of slope (semi)stable pure dimension one sheaves F on Y with numerical invariants

$$\mathrm{ch}_2(F)=\beta,\qquad \chi(F)=n.$$

Suppose furthermore that (β,n) are primitive, such that there are no strictly semistable points. If $\mathcal{M}(Y,\beta,n)$ is smooth, the BPS states are in one-to-one correspondence with cohomology classes of the moduli space. This identification still holds [15] when $\mathcal{M}(Y,\beta,n)$ is a singular variety provided that singular cohomology is replaced by intersection cohomology. In such cases, there is a geometric construction of the expected $SL(2)_L \times SL(2)_R$ action on the BPS Hilbert space if in addition there is a natural Hitchin map $h : \mathcal{M}(Y, \beta, n) \to \mathcal{B}$ to a smooth projective variety $\mathcal{B}$. Then [15] the action follows from decomposition theorem [2], [6] and as well as the relative hard Lefschetz theorem [7]. In particular the action of the positive roots J_L^+, J_R^+ should is given by cup product with a relative ample class ω_h, respectively the pull back of an ample class $\omega_{\mathcal{B}}$ on the base. One then obtains a decomposition of intersection cohomology of the form

$$IH^*(\mathcal{M}(Y, \beta, n)) \simeq \bigoplus_{(j_L, j_R) \in \mathbb{Z}^2} R(j_L, j_R)^{\oplus d(j_L, j_R)}$$

where $R(j_L, j_R)$ is the irreducible representation of $SL(2)_L \times SL(2)_R$ with highest weight (j_L, j_R). A priori the multiplicities $d(j_L, j_R)$ should depend on n for a fixed curve class β. Since no such dependence is observed in the low energy theory, one is lead to further conjecture that the $d(j_L, j_R)$ are in fact independent of n, as long as the numerical invariants (β, n) are primitive. Granting this additional conjecture, the refined BPS invariants are given by $N(Y, \beta, j_L, j_R) = d(j_L, j_R)$.

In more general situations no rigorous mathematical construction of a BPS cohomology theory is known. There is however a rigorous construction of unrefined GV invariants via stable pairs [30, 31] which will be briefly reviewed shortly. It is worth noting that the BPS cohomology theory would have to detect the scheme structure and the obstruction theory of the moduli space as is the case in [30, 31].

Concrete examples where the moduli space $\mathcal{M}(Y, \beta, n)$ is smooth are usually encountered in local models, in which case Y is a noncompact threefold.

Example 1.1. *Let S be a smooth Fano surface and Y the total space of the canonical bundle K_S. Then any semistable pure dimension one sheaf F must be scheme theoretically supported on the zero section. Therefore there is a natural Hitchin map to a linear system on S. For primitive numerical invariants (β, n), the moduli space is smooth and the Hitchin map is projective.*

Example 1.2. *Let X be a smooth projective curve and D an effective divisor on X, possibly trivial. Let Y be the total space of the rank two bundle $\mathcal{O}_X(-D) \oplus K_X(D)$. Note that $H_2(Y) \simeq \mathbb{Z}$ is generated by the class σ of the 0 section. Let $\mathcal{M}(Y, d, n)$ be the moduli space of stable pure dimension sheaves F on Y with compact support and numerical invariants*

$$\mathrm{ch}_2(F) = d\sigma, \qquad \chi(F) = n.$$

Let $\mathcal{X}_r^e(X)$ be the moduli space of rank $r \geq 1$, degree $e \in \mathbb{Z}$ stable Hitchin pairs on X. Then it is easy to prove the following statements.

a) If $D = 0$ and $(d, n) = 1$, there is an isomorphism

$$\mathcal{M}(Y, d, n) \simeq \mathcal{H}_d^{n+d(g-1)}(X) \times \mathbb{C}.$$

b) If $D \neq 0$ and $(d, n) = 1$, there is an isomorphism

$$\mathcal{M}(Y, d, n) \simeq \mathcal{H}_d^{n+d(g-1)}(X).$$

As mentioned above unrefined GV numbers can be defined via stable pair or Donaldson-Thomas invariants. From a string theoretic point of view, this has been explained in [8] using IIA/M-theory duality. Let $\mathcal{Z}_{DT}(Y, q, Q)$ be the (unrefined) reduced Donaldson-Thomas theory of Y defined in [24], or, equivalently, the stable pair theory of Y defined in [30]. In a IIA compactification on Y, $\mathcal{Z}_{DT}(Y, q, Q)$ is the generating function for the degeneracies of BPS states corresponding to bound states of one D6-brane and arbitrary D2-D0 brane configurations on Y. According to [12, 8], M-theory/IIA duality yields an alternative expression for this generating function in terms of the five dimensional BPS indices $N(Y, \beta, j_L)$. Then

$$\mathcal{Z}_{DT}(Y, q, Q) = \exp\left(F_{GV}(Y, q, Q)\right), \tag{1.1}$$

where

$$F_{GV}(Y, q, Q) = \sum_{k\geq 1} \sum_{\beta\in H_2(Y),\beta\neq 0} \sum_{j_L\in \frac{1}{2}\mathbb{Z}} \frac{Q^{k\beta}}{k}(-1)^{2j_L} N(Y, \beta, j_L) \frac{q^{-2kj_L} + \cdots + q^{2kj_L}}{(q^{k/2} - q^{-k/2})^2}. \tag{1.2}$$

Relation (1.1) can be either inferred from [12] relying on the GW/DT correspondence conjectured in [24], or directly derived on physical grounds from Type IIA/M-theory duality [8].

According to [18], a similar relation is expected to hold between refined stable pair invariants and the GV numbers $N(Y, \beta, j_L, j_R)$. As explained in [9] refined stable pair invariants are obtained as a specialization of the virtual motivic invariants of Kontsevich and Soibelman [21]. Then one expects [18] a relation of the form

$$\mathcal{Z}_{DT,Y}(q, Q, y) = \exp\left(F_{GV,Y}(q, Q, y)\right), \tag{1.3}$$

where

$$\begin{aligned} F_{GV,Y}(q, Q, y) = &\sum_{k\geq 1} \sum_{\beta\in H_2(Y),\beta\neq 0} \sum_{j_L,j_R\in \frac{1}{2}\mathbb{Z}} \frac{Q^{k\beta}}{k}(-1)^{2j_L+2j_R} N(Y, \beta, j_L, j_R) \\ &q^{-k}\frac{(q^{-2kj_L} + \cdots + q^{2kj_L})(y^{-2kj_R} + \cdots + y^{2kj_R})}{(1-(qy)^{-k})(1-(qy^{-1})^{-k})}. \end{aligned} \tag{1.4}$$

The expression (1.4) was written in [18] in different variables, $(q^{-1}y, q^{-1}y^{-1})$.

The main goal of this note is to point out that the refined GV expansion (1.3) for a local curve geometry is related via a simple change of variables to the Hausel-Rodriguez-Villegas formula for character varieties. There a few conjectural steps involved in this identification. First, it relies on a explicit conjectural formula for the refined stable pair theory of a local curve derived in section (3) from geometric engineering and instanton sums. In fact, it is expected that a rigorous construction of motivic stable pair theory of local curves should be possible following the program of Kontsevich and Soibelman [21]. A conjectural motivic formula generalizing equation (3.4) has been recently written down by Mozgovoy [27]. Second, as explained in detail in section (4), the refined GV invariants of the local curve are in fact perverse Betti numbers of the Hitchin moduli space. Therefore, the conversion of the HRV formula into a refined GV expansion relies on the identification between the weight filtration on the cohomology of character varieties and the perverse filtration on the cohomology of the Hitchin system conjectured by de Cataldo, Hausel and Migliorini [5]. This will be referred to as the $P = W$ conjecture. From a physicist's perspective, the connection found here provides a natural explanation as well as strong evidence for this conjecture. Finally, note that further evidence for all the claims of the present paper comes from the recent rigorous results of [27, 25, 26]. In [27] it is rigorously proven that the refined theory of the local curve implies the HRV conjecture for the Poincaré polynomial of the Hitchin system via motivic wallcrossing while [25, 26] prove expansion formulas analogous to (1.3) for families of irreducible reduced plane curves.

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2. Hausel-Rodriguez-Villegas formula and $P = W$

Let X be a smooth projective curve over $\mathbb{C}$ of genus $g \geq 1$, and $p \in X$ an arbitrary closed point. Let $\gamma_p \in \pi_1(X \setminus \{p\})$ be the natural generator associated to p. For any coprime integers $r \in \mathbb{Z}_{\geq 1}$, $e \in \mathbb{Z}$, the character variety $\mathcal{C}^e_r(X)$ is the moduli space of representations

$$\phi : \pi_1(X \setminus \{p\}) \to GL(r, \mathbb{C}), \qquad \phi(\gamma_p) = e^{2i\pi e/r} I_r$$

modulo conjugation. $\mathcal{C}^e_r(X)$ is a smooth quasi-projective variety, and its rational cohomology $H^*(\mathcal{C}^e_r(X))$ carries a mixed Hodge structure

$$W^k_0 \subset \cdots W^k_i \subset \cdots \subset W^k_{2k} = H^k(\mathcal{C}^n_r(X)). \tag{2.1}$$

According to [13], $W^k_{2i} = W^k_{2i+1}$ for all $i = 0, \ldots, 2k$, hence one can define the virtual Poincaré polynomial

$$W(\mathcal{C}^e_r(X), z, t) = \sum_{i,k} \dim(W^k_i / W^k_{i-1}) t^k z^{i/2}. \tag{2.2}$$

Moreove the virtual Poincaré polynomial is independent on e for fixed r, therefore it will be denoted below by $W_r(z, t)$. Obviously $W_r(1, t)$ is the usual Poincaré polynomial. The opposite specialization, $P_r(X, z, 1)$ is identified with the E-polynomial with compact support via Poincaré duality

$$H^k_c(\mathcal{C}^n_r(X)) \times H^{2d-k}(\mathcal{C}^n_r(X)) \to \mathbb{C}.$$

Using number theoretic considerations Hausel-Rodriguez-Villegas [13] derive a conjectural formula for the Poincaré polynomials $W_r(z, t)$ as follows.

2.1. **Hausel-Rodriguez-Villegas formula.** The conjecture formulated in [13] expresses the generating function

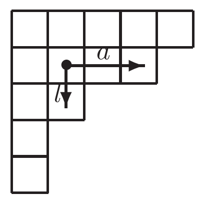
$$F_{HRV}(z, t, T) = \sum_{r,k \geq 1} B_r(z^k, t^k) W_r(z^k, t^k) \frac{T^{kr}}{k},$$

$$B_r(z, t) = \frac{(zt^2)^{(1-g)r(r-1)}}{(1-z)(1-zt^2)},$$

as

$$F_{HRV}(z, t, T) = \ln Z_{HRV}(z, t, T) \tag{2.3}$$

where $Z_{HRV}(z,t,T)$ is a sum of rational functions associated to Young diagrams. Given a Young diagram μ as shown below



let μ_i be the length of the i-th row, $|\mu|$ the total number of boxes of μ, and μ^t the transpose of μ. For any box $\Box = (i,j) \in \mu$ let

$$a(\Box) = \mu_i - j, \qquad l(\Box) = \mu^t_j - i, \qquad h(\Box) = a(\Box) + l(\Box) + 1,$$

be the arm, leg, respectively hook length. Then

$$\mathcal{Z}_{HRV}(z,t,T) = \sum_{\mu} \mathcal{H}^{\mu}_g(z,t) T^{|\mu|}$$

where

$$\mathcal{H}^{\mu}_g(z,t) = \prod_{\Box\in\mu} \frac{(zt^2)^{l(\Box)(2-2g)}(1 - z^{h(\Box)}t^{2l(\Box)+1})^{2g}}{(1 - z^{h(\Box)}t^{2l(\Box)+2})(1 - z^{h(\Box)}t^{2l(\Box)})}.$$

The main observation in this note is that equation (2.3) can be identified with the expansion of the refined Donaldson-Thomas series of a certain Calabi-Yau threefold in terms of numbers of BPS states.

2.2. **Hitchin system and $P = W$.** Let $\mathcal{H}^e_r(X)$ be the moduli space of stable Higgs bundles (E, Φ) on X, where Φ is a Higgs field with coefficients in K_X. For coprime (r, e) this is a smooth quasi-projective variety equipped with a projective Hitchin map

$$h : \mathcal{H}^e_r(X) \to \mathcal{B}$$

to the affine variety

$$\mathcal{B} = \oplus_{i=1}^r H^0(K_X^{\otimes i}).$$

The decomposition of the direct image $Rh_*\underline{\mathbb{Q}}$ into perverse sheaves yields [6, 5] a perverse filtration

$$0 = P^k_0 \subset P^k_1 \subset \cdots \subset P^k_k = H^k(\mathcal{H}^e_r(X))$$

on cohomology. It is well known that $\mathcal{C}^e_r(X)$ and $\mathcal{H}^e_r(X)$ are identical as smooth real manifolds. More precisely the complex structures are related by a hyper-Kähler rotation. Therefore there is a natural identification $H^*(\mathcal{C}^e_r(X)) = H^*(\mathcal{H}^e_r(X))$. Then it is conjectured in [5] that the two filtrations W^k_j, P^k_j coincide,

$$W^k_{2j} = P^k_j$$

for all k, j. This is proven in [5] for Hitchin systems of rank $r = 2$.

For future reference note that an h-relatively ample class ω yields an hard Lefschetz isomorphism [7]

$$\omega^l : Gr^P_{d-l} H^k \xrightarrow{\sim} Gr^P_{d+l} H^{k+2l}.$$

This is known under the name of relative hard Lefschetz theorem. In particular, one then obtains a splitting of the perverse filtration.

Note that granting the $P = W$ conjecture, equation (2.3) yields explicit formulas for the perverse Poincaré polynomial of the Hitchin moduli space. In particular, by specialization to $z = 1$ it determines the Poincaré polynomial of the Hitchin moduli space of any rank $r \geq 1$.

3. Refined stable pair invariants of local curves

Let Y be the total space of the rank two bundle $\mathcal{O}_X(-D) \oplus K_X(D)$ where D is an effective divisor of degree $p \geq 0$ on X as in Example (1.2). Note that $H_2(Y) \simeq \mathbb{Z}$ is generated by the class σ of the zero section. Following [30], stable pairs on Y are two term complexes $P = (\mathcal{O}_Y \xrightarrow{s} F)$ where F is a pure dimension one sheaf and s a generically surjective section. Since Y is noncompact, in the present case, it will be also required that F have compact support, which must be necessarily a finite cover of X. The numerical invariants of F will be

$$\mathrm{ch}_2(F) = d\sigma, \qquad \chi(F) = n.$$

Then according to [30], there is a quasi-projective fine moduli space $\mathcal{P}(Y, d, n)$ of pairs of type (d, n) equipped with a symmetric perfect obstruction theory. The moduli space also carries a torus action induced by the $\mathbb{C}^\times$ action on Y which scales $\mathcal{O}_X(-D), K_X(D)$ with weights $-1, 1$. Virtual numbers of pairs can be defined by equivariant virtual integration. According to [1], the resulting invariants coincide with the Behrend Euler numbers of the moduli spaces,

$$P(d, n) = \chi^B(\mathcal{P}(Y, d, n)).$$

Let

$$Z_{PT}(Y, q, Q) = 1 + \sum_{d\geq 1} \sum_{n\in\mathbb{Z}} P(d, n) Q^d q^n.$$

Applying the motivic Donaldson-Thomas formalism of Kontsevich and Soibelman, one obtains a refinement $P^{ref}(d, n, y)$ of stable pair invariants modulo foundational issues. The $P^{ref}(d, n, y)$ are Laurent polynomials of the formal variable y with integral coefficients. In a string theory compactification on Y these coefficients are numbers of D6-D2-D0 bound states with given four dimensional spin quantum number. The resulting generating series will be denoted by $Z^{ref}_{PT}(Y, q, Q, y)$.

3.1. **TQFT formalism.** A TQFT formalism for unrefined Donaldson-Thomas theory of a local curve has been developed in [29], in parallel with a similar construction [3] in Gromov-Witten theory. Very briefly, the final result is that the generating series of local invariants is obtained by gluing vertices corresponding to a pair of pants decomposition of the Riemann surface X. Each such vertex is a rational function $P_{\mu_i}(q)$ labelled by three partitions μ_i, $i = 1, 2, 3$ corresponding to the three boundary components. In the equivariant Calabi-Yau case a nontrivial result is obtained only for identical partitions, $\mu_i = \mu$, $i = 1, 2, 3$, in which case

$$P_\mu(q) = \prod_{\Box\in\mu} (q^{h(\Box)/2} - q^{-h(\Box)/2}).$$

Then the generating function is given by

$$Z_{DT}(Y, q, Q) = \sum_{\mu} (-1)^{p|\mu|} q^{-(g-1-p)\kappa(\mu)} (P_\mu(q))^{2g-2} Q^{|\mu|} \tag{3.1}$$

where

$$\kappa(\mu) = \sum_{\Box\in\mu}(i(\Box) - j(\Box)).$$

3.2. **Refined invariants from instanton sums.** Although the refined stable pair invariants are not rigorously constructed for higher genus local curves, string duality leads to an explicit conjectural formula for the series $Z^{ref}_{PT}(Y, q, Q, y)$. This follows using geometric engineering [19] of supersymmetric five dimensional gauge theories. In the present case, the threefold Y yields a $U(1)$ gauge theory with g adjoint hypermultiplets. By analogy with previous toric models [22, 10, 28, 16, 17, 11, 14, 20, 23, 18], it is expected that the refined stable pair invariants of invariants of Y will be completely determined by the instanton partition function of this theory [28]. The instanton partition function of $U(1)$ five dimensional gauge theory with g adjoint hypermultiplets is constructed in terms of quivariant K-theoretic invariants of the Hilbert scheme of points in $\mathbb{C}^2$ as follows [4].

Let $\mathcal{H}ilb^d(\mathbb{C}^2)$ denote the Hilbert scheme of length $d \geq 1$ zero dimensional subschemes of $\mathbb{C}^2$. It is smooth, quasi-projective and carries a natural tautological vector bundle $\mathcal{V}_d$ whose fiber at a point $[Z]$ is the space of global sections $H^0(\mathcal{O}_Z)$. For each $d \geq 1$ let

$$\mathcal{E}_d = T^*\mathcal{H}ilb^d(\mathbb{C}^2)^{\oplus d} \otimes \det(\mathcal{V}_d)^{-1}.$$

Note that $\mathcal{E}_d$ carries a natural equivariant structure with respect to the $\mathbb{C}^\times \times \mathbb{C}^\times$ induced by the scaling action on $\mathbb{C}^2$. Let q_1, q_2 be the characters of the basic one dimensional representations of $\mathbb{C}^\times \times \mathbb{C}^\times$. Then the equivariant K-theoretic partition function is defined by

$$Z_{inst}(q_1, q_2, \widetilde{Q}, \widetilde{y}) = \sum_{k\geq 0} \chi_{\widetilde{y}}(T^*\mathcal{H}ilb_k(\mathbb{C}^2)^{\oplus g} \otimes \det(\mathcal{V}_d)^{1-g-p})\widetilde{Q}^k \tag{3.2}$$

where

$$\chi_{\widetilde{y}}(\mathcal{E}_d) = \sum_{i,j}(-\widetilde{y})^j(-1)^i \mathrm{ch}\, H^i(\wedge^j\mathcal{E}_d)$$

is the equivariant $\chi_{\widetilde{y}}$-genus of $\mathcal{E}_d$. A fixed point theorem gives an explicit formula for $Z_{inst}(q_1, q_2, \widetilde{Q}, \widetilde{y})$ as a sum over partitions.

$$Z_{inst}(q_1, q_2, \widetilde{Q}, \widetilde{y}) = \sum_{\mu}\prod_{\Box\in\mu}(q_1^{-l(\Box)}q_2^{-a(\Box)})^{g-1+p}$$
$$\frac{(1-\widetilde{y}q_1^{-l(\Box)}q_2^{a(\Box)+1})^g(1-\widetilde{y}q_1^{l(\Box)+1}q_2^{-a(\Box)})^g}{(1-q_1^{-l(\Box)}q_2^{a(\Box)+1})(1-q_1^{l(\Box)+1}q_2^{-a(\Box)})}\widetilde{Q}^{|\mu|}$$

The resulting conjectural expression for the refined stable pair partition function is then [4]

$$Z^{ref}_{PT}(Y, q, Q, y) = Z_{inst}(q^{-1}y, qy, (-1)^{g-1}y^{2-g}Q, y^{-1}). \tag{3.3}$$

A straightforward computation shows that

$$Z^{ref}_{PT}(Y, q, Q, y) = \sum_{\mu}\Omega^{\mu}_{g,p}(q, y)Q^{|\mu|} \tag{3.4}$$

where

$$\Omega_g^\mu(q,y) = (-1)^{p|\mu|} \prod_{\Box\in\mu} \left[\left(q^{l(\Box)-a(\Box)} y^{-(l(\Box)+a(\Box))}\right)^p (qy^{-1})^{(2l(\Box)+1)(g-1)} \right.$$
$$\left. \frac{(1-q^{-h(\Box)}y^{l(\Box)-a(\Box)})^{2g}}{(1-q^{-h(\Box)}y^{l(\Box)-a(\Box)-1})(1-q^{-h(\Box)}y^{l(\Box)-a(\Box)+1})} \right].$$

The change of variables in (3.3) does not have a conceptual derivation. This conjecture is supported by extensive numerical computations involving wallcrossing for refined invariants in [4]. Further supporting evidence for the formula (3.3) is obtained by comparison with the unrefined TQFT formula (3.1) for local curves. Specializing the right hand side of (3.3) at $y=1$, one obtains

$$Z_{PT}^{ref}(Y,q,Q,1) = \sum_\mu Q^{|\mu|} \prod_{\Box\in\mu} (-1)^{p|\mu|} q^{(g-1+p)(l(\Box)-a(\Box))} (q^{h(\Box)/2} - q^{-h(\Box)/2})^{2g-2}.$$

Agreement with (3.1) follows from the identity

$$\sum_{\Box\in\mu} (l(\Box)-a(\Box)) = \sum_{\Box\in\mu} (j(\Box)-i(\Box)) = -\kappa(\mu).$$

Finally, note that the expression (3.3) with $p=0$ is related to the left hand side of the HRV formula by

$$Z_{HRV}(z,t,T) = Z_{PT}^{ref}(Y,(zt)^{-1},(zt^2)^{g-1}T,t). \tag{3.5}$$

4. HRV formula as a refined GV expansion

This section spells out in detail the construction of refined GV invariants of a local curve geometry as in Example (1.2) with $p=\deg(D)=0$ in terms of the perverse filtration on the cohomology of the Hitchin moduli space. Using the conjectural formula (3.3), it will be shown that equation (1.3) yields the HRV formula by a monomial change of variables. As observed in Example (1.2), the moduli space of slope stable pure dimension one sheaves F on Y with compact support and numerical invariants

$$\mathrm{ch}_2(F) = r\sigma, \qquad \chi(F) = n$$

is isomorphic to $\mathbb{C}\times\mathcal{H}_r^{n+r(g-1)}(X)$ provided that $(r,n)=1$. Therefore, following the general arguments in the introduction, one should be able to define refined GV invariants using the decomposition theorem for the Hitchin map $h:\mathcal{H}_r^e(X)\to\mathcal{B}$, $e=n+r(g-1)$. However, since the base of the Hitchin fibration is a linear space, there will not exist an $SL(2)_L\times SL(2)_R$ action on cohomology as required by M-theory. In this situation one can only define an $SL(2)_L\times\mathbb{C}_R^\times$-action where $\mathbb{C}_R^\times$ can be thought of as a Cartan subgroup of $SL(2)_R$. This action can be explicitly described in terms of the perverse sheaf filtration constructed in [5, Sect. 1.3]. Note that a relative ample class ω_h for the Hitchin map yields an hard Lefschetz isomorphism [7]

$$\omega^l : Gr_{d-l}^P H^k(\mathcal{H}_r^e(X)) \xrightarrow{\sim} Gr_{d+l}^P H^{k+2l}(\mathcal{H}_r^e(X)).$$

This yields a splitting

$$H^k(\mathcal{H}_X(r,e)) \simeq \bigoplus_p H_p^k$$

of the perverse filtration, where

$$H_p^k \simeq \bigoplus_{i+2j=p} Q^{i,j;k}, \qquad Q^{i,j;k} = \omega_h^j Q^{i,0;k-2j}.$$

Let $Q^{i,j} = \bigoplus_{k\geq 0} Q^{i,j;k}$. By construction, for fixed $0 \leq i \leq d$, there is an isomorphism

$$\bigoplus_{j=0}^{d-i} Q^{i,j} \simeq R_{(d-i)/2}^{\oplus \dim(Q^{i,0})}$$

where R_{j_L} is the irreducible representation of $SL(2)_L$ with spin $j_L \in \frac{1}{2}\mathbb{Z}$. The generator J_L^+ is represented by cup-product with ω_h, and the $Q^{i,j}$ is an eigenspace of the Cartan generator J_L^3 with eigenvalue $j - (d-i)/2$. Note that cup-product with ω_h preserves the grading $k-d-2j$ therefore one can define an extra $\mathbb{C}^\times$-action on $H^*(\mathcal{H}_r^e(X))$ which scales $Q^{i,j;k}$ with weight $d + 2j - k$. This torus action will be denoted by $\mathbb{C}_R^\times \times H^*(\mathcal{H}_r^e(X)) \to H^*(\mathcal{H}_r^e(X))$. Note also that

$$d + 2j - k \geq -d$$

since $j \geq 0$ and $k \leq -2d$.

In conclusion, in the present local curve geometry the $SL(2)_L \times SL(2)_R$ action on the cohomology of the moduli space of D2-D0 branes is replaced by an $SL(2)_L \times \mathbb{C}_R^\times$ action. This is is certainly puzzling from a physical perspective since the BPS states are expected to form five-dimensional spin multiplets. The absence of a manifest $SL(2)_R$ symmetry of the local BPS spectrum is due to noncompactness of the moduli space. This is simply a symptom of the fact that there is no well defined physical decoupling limit associated to a local higher genus curve as considered here in M-theory. In principle, in order to obtain a physically sensible theory, one would have to construct a Calabi-Yau threefold $\overline{Y}$ containing a curve X with infinitesimal neighborhood isomorphic to Y so that the moduli space $\mathcal{M}_{\overline{Y}}(r[X], n)$ is compact and there is an embedding $H^*(\mathcal{M}_Y(r, n)) \subset H^*(\mathcal{M}_{\overline{Y}}(r[X], n))$. The cohomology classes in the complement would then provide the missing components of the five-dimensional spin multiplets. Such a construction seems to be very difficult, and it is not in fact needed for the purpose of the present paper.

Given the $SL(2)_L \times \mathbb{C}_R^\times$ action described in the previous paragraph, one can define the following local version of the refined Gopakumar-Vafa expansion (1.4).

$$\begin{aligned} F_{GV,Y}(q, Q, y) = \sum_{k\geq 1} \sum_{r\geq 1} \sum_{j_L=0}^{d/2} \sum_{l\geq -d} & \frac{Q^{kr}}{k} (-1)^{2j_L+l} N_r((j_L, l)) \\ & \frac{q^{-k}(q^{-2kj_L} + \cdots + q^{2kj_L}) y^{kl}}{(1-(qy)^{-k})(1-(qy^{-1})^{-k})}. \end{aligned} \tag{4.1}$$

where

$$N_r(j_L, l) = \dim(Q^{d-2j_L,0;d+l}).$$

Making the same change of variables as in equation (3.5) yields

$$F_{GV,Y}((zt)^{-1}, (zt^2)^{g-1}T, t) = \sum_{k\geq 1} \sum_{r\geq 1} \frac{T^{kr}}{k} B_r(z^k, t^k) P_r(z^k, t^k) \tag{4.2}$$

where $B_r(z,t)$ is defined above equation (2.3), and

$$P_r(z,t) = \sum_{j=0}^{d} \sum_{l\geq 0} (-1)^{j+l} N_r((j-d)/2, l-d)\, t^l\, (1 + \cdots + (zt)^{2j}).$$

Now it is clear that the change of variables

$$(q, Q, y) = ((zt)^{-1}, (zt^2)^{g-1}T, t)$$

identifies the HRV formula (2.3) with the refined GV expansion (1.3) for a local curve provided that

$$P_r(z,t) = W_r(z,t). \tag{4.3}$$

However, given the cohomological definition of the refined GV invariants $N_r(j_L, l)$, relation (4.2) follows from the $P = W$ conjecture of [5]. This provides a string theoretic explanation as well as strong evidence for this conjecture.

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[1] Department of Mathematics, National Taiwan University, Taipei, Taiwan

[2] NHETC, Rutgers University, Piscataway, NJ 08854-0849 USA