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Abstract

In this paper, we introduce the concept of k-power domination which is a common generalization of domination and power domination. We extend several known results for power domination to k-power domination. Concerning the complexity of the k-power domination problem, we first show that deciding whether a graph admits a k-power dominating set of size at most t is NP-complete for chordal graphs and for bipartite graphs. We then give a linear algorithm for the problem on trees. Finally, we propose sharp upper bounds for the power domination number of connected graphs and of connected claw-free $(k+2)$-regular graphs.

Keywords: Power domination, electrical network monitoring, domination

1. Introduction

In this paper we only consider simple graphs, that are graphs without multiple edges or loops. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use the following definitions.

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The *open neighbourhood of* a vertex v, denoted $N_G(v)$, is the set of vertices adjacent to v, namely $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The *closed neighbourhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The *open* (resp. *closed*) *neighbourhood* of a set $S \subseteq V$ is the union of the open (resp. closed) neighbourhoods of its elements: $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = \bigcup_{v \in S} N_G[v] = N_G(S) \cup S$. When G is clear from context, we use N instead of N_G. The *degree* of a vertex v, denoted $d(v)$, is the size of its open neighbourhood $|N(v)|$. The maximum degree of the graph G is denoted by $\Delta(G)$.

A *dominating set* of a graph G is a set of vertices S such that $N[S] = V(G)$. The *domination number* of a graph G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. The *dominating set problem*, that is deciding whether a graph admits a dominating set of size at most t, is NP-complete [4, 7] (even restricted to bipartite or chordal graphs).

Power domination was introduced in [3, 13] to model a problem of monitoring electrical networks. It was then described as a graph theoretical problem in [10]. The problem is similar to a problem of domination, in which, additionally, the possibility of some propagation according to Kirschoff laws is considered. The definition of power domination, originally asking to monitor both edges and vertices, was simplified to the following, introduced independently in [5, 6].

Let G be a graph and S a subset of its vertices. The *set monitored by* S, denoted by $M(S)$, is defined algorithmically as follows:

- (*domination*) $M(S) \leftarrow S \cup N(S)$,
- (*propagation*) as long as there exists $v \in M(S)$ such that $N(v) \cap (V(G) - M(S)) = \{w\}$, set $M(S) \leftarrow M(S) \cup \{w\}$.

In other words, the set $M(S)$ is obtained from S as follows. First put into $M(S)$ the vertices from the closed neighbourhood of S. Then repeatedly add to $M(S)$ vertices w that have a neighbour v in $M(S)$ such that all the other neighbours of v are already in $M(S)$. After no such vertex w exists, the set monitored by S has been constructed. The set S is called a *power dominating set* of G if $M(S) = V(G)$ and the *power domination number* $\gamma_{\mathrm{P}}(G)$ is the minimum cardinality of a power dominating set.

This definition implies some propagating behaviour of the set of monitored vertices, a phenomenon very different from the standard domination parameter. Different works were started on the topic. From the algorithmic point of view, the power domination problem was known to be NP-complete

[1, 2, 8, 9, 10] and approximation algorithms were given in [2]. On the other hand, linear-time algorithms for the power domination problem were given for trees [10], for interval graphs [12] and for block graphs [15]. Parameterized results were given in [11]. The exact values for the power domination numbers are determined for various products of graphs [5, 6]. Bounds for the power domination numbers of connected graphs and of claw-free cubic graphs are given in [17] and for planar or outerplanar graphs with bounded diameter in [16].

Let k be a nonnegative integer. Let us introduce k-power domination, using a definition of monitored set close to what Aazami proposed in [1].

Definition 1 (Monitored set). *Let G be a graph and $S \subseteq V(G)$. We define the sets $\left(\mathcal{P}_G^i(S)\right)_{i\geq 0}$ of vertices* monitored *by S at step i by the following rules.*

- $\mathcal{P}_G^0(S) = N[S]$.
- $\mathcal{P}_G^{i+1}(S) = \cup\{N[v]\colon v \in \mathcal{P}_G^i(S)$ *such that* $\left|N[v] \setminus \mathcal{P}_G^i(S)\right| \leq k\}$.

Remark that $\mathcal{P}_G^i(S) \subseteq \mathcal{P}_G^{i+1}(S) \subseteq V(G)$ for any i. Moreover, every time a vertex of the set $\mathcal{P}_G^i(S)$ has at most k neighbours outside the set, we add its neighbours to the next generation $\mathcal{P}_G^{i+1}(S)$.

If $\mathcal{P}_G^{i_0}(S) = \mathcal{P}_G^{i_0+1}(S)$ for some i_0, then $\mathcal{P}_G^j(S) = \mathcal{P}_G^{i_0}(S)$ for any $j \geq i_0$. We thus define $\mathcal{P}_G^\infty(S) = \mathcal{P}_G^{i_0}(S)$.

Definition 2 (k-power dominating set). *A set S such that $\mathcal{P}_G^\infty(S) = V(G)$ is a k-*power dominating set *of G. The least cardinality of such a set is called the k-*power domination number *of G, written $\gamma_{\mathrm{P},k}(G)$. A $\gamma_{\mathrm{P},k}(G)$-*set *is a minimum k-power dominating set of G.*

When G is clear from context, we will use $\mathcal{P}^i(S)$ to denote $\mathcal{P}_G^i(S)$. From this definition, the following observation clearly holds.

Observation 3. *Let G be a graph and $S, S' \subseteq V(G)$. If $\mathcal{P}^i(S) \subseteq \mathcal{P}^j(S')$ for some integers i and j, then $\mathcal{P}^{i+1}(S) \subseteq \mathcal{P}^{j+1}(S')$ and so $\mathcal{P}^\infty(S) \subseteq \mathcal{P}^\infty(S')$ by extension. In particular, if S is a k-power dominating set of G, then so is S'.*

We introduce k-power domination as a natural generalization of power domination, with correspondence when $k = 1$. It also generalizes usual domination, for $k = 0$. Whereas power domination seemed a problem of very different type than domination, it is remarkable how this generalization unifies the two problems. First, the algorithm we propose here for computing a k-power dominating set of a tree is closely related to the algorithm proposed for domination in [14], whereas the algorithm proposed for power domination in [10] was identifying *spiders* in the tree, a very different technique. Moreover, the bound in Theorem 11 is a very natural generalization of the original classical bound for domination in graphs with no isolated vertices, and the examples of graphs reaching the bound are similar to the coronas, which are graph reaching the bound for domination. Similarly, Lemma 10 is a non trivial generalization of the fact that a dominating set in a graph is minimal only if each of its vertices has a private neighour.

These similarities make generalized power domination shed a new light on power domination. It seems that one might use this new approach to extend some of the numerous results known for domination to the applied problem of power domination.

The paper is organized as follows. Section 2 is dedicated to complexity results; we show that the k-power domination number is an NP-complete problem even in the case of chordals graphs or bipartite graphs and we give a linear time algorithm computing $\gamma_{\mathrm{P},k}(G)$ when G is a tree.

In Section 3, we prove that, for any connected graph G of order n, $\gamma_{\mathrm{P},k}(G) \leq \frac{n}{k+2}$, and we show the sharpness of the bound. Also we show that, for any claw-free $(k+2)$-regular graph of order n, $\gamma_{\mathrm{P},k}(G) \leq \frac{n}{k+3}$, and we characterize the graphs achieving this last bound.

2. Complexity results

For any graph G and any nonnegative integer k, let G_k be the graph obtained from G by adding k new neighbours to each vertex of G. By the construction of G_k, we get $\gamma_{\mathrm{P},k}(G_k) = \gamma(G)$. Notice that if G is chordal (or bipartite), then so is G_k. The NP-completeness of the domination problem [7] gives the following result.

Theorem 4. *The k-power domination problem, that is deciding whether a graph admits a k-power dominating set of size at most t, is NP-complete for chordal graphs and for bipartite graphs.*

A linear algorithm for trees

We now give a linear-time algorithm for the k-power domination problem on trees. We in fact solve a slightly more general problem.

The algorithm we propose is largely inspired by the algorithm proposed in [14] for domination. The main idea is to put labels on the vertices, then iteratively deleting the leafs, recording all useful information about the deleted subtree within the labels on the parent vertex.

We use two labels for each vertex v, denoted a_v and b_v. The first label a_v conveys the same partition idea as in [14]. The vertex set of the graph is partitioned into three sets $\mathcal{R}, \mathcal{B}$ and $\mathcal{F}$ corresponding to labels R, B and F. We assign a vertex the label R (for *required*) when the optimal choice is to put the corresponding vertex in the k-power dominating set, B (for *bound*) when the vertex is not yet monitored in any way, and F (for *free*) when the vertex is already monitored.

The second label b_v is an integer, useful only for states F and B. When a vertex gets monitored, it may be used for propagation, but to at most k neighbours. This second label b_v denotes how many neighbours the corresponding vertex may still propagate to. This label may only decrease one by one when we require a propagation from the corresponding vertex.

Formally, to each vertex v of G we associate a label $L(v) = (a_v, b_v)$ where $a_v \in \{\mathrm{B}, \mathrm{F}, \mathrm{R}\}$ and $b_v \in \{0, 1, \ldots, k\}$. An L-power dominating set of $G = (V, E)$ is a subset S of V such that $v \in S$ for all vertices v with $a_v = \mathrm{R}$, where the sets $\left(\mathcal{P}^i_{G,L}(S)\right)_{i \geq 0}$ of vertices monitored by S at step i are defined by the following rules.

- $\mathcal{P}^0_{G,L}(S) = N[S] \cup \{v \colon a_v = \mathrm{F}\}$.
- $\mathcal{P}^{i+1}_{G,L}(S) = \mathcal{P}^i_{G,L}(S) \cup \{N[v] \colon v \in \mathcal{P}^i_{G,L}(S) \text{ such that } |N[v] \setminus \mathcal{P}^i_{G,L}(S)| \leq b_v\}$.

When $v \in \mathcal{P}^i_{G,L}(S)$ satisfies $|N[v] \setminus \mathcal{P}^i_{G,L}(S)| \leq b_v$, we say that v satisfies the *L-propagating condition* (for short L-PC). The *L-power domination number* of G, denoted $\gamma_{\mathrm{P},L}(G)$, is defined as the least cardinality of a L-power dominating set of G. A *$\gamma_{\mathrm{P},L}(G)$-set* denotes a minimum L-power dominating set of G.

Remark that when all $L(v) = (\mathrm{B}, k)$, L-power domination is the same as the k-power domination. Having this general setting in mind, we now have the following theorem from which one can infer a linear-time algorithm for computing a minimum L-power dominating set of a tree.

Theorem 5. *Consider a graph $G = (V, E)$ and a vertex labeling L. Suppose x is a vertex adjacent to only one other vertex y. If $G' = G - x$ and L' is the restriction of L on $V' = V \setminus \{x\}$ with modification on $L'(y)$ as indicated below, then the following statements hold.*

(1) *If $a_x = \mathrm{R}$, then $\gamma_{\mathrm{P},L}(G) = \gamma_{\mathrm{P},L'}(G') + 1$, where a'_y is redefined to be* F *when $a_y = \mathrm{B}$.*

Otherwise, we assume $a_x \neq \mathrm{R}$ for (2) to (5).

(2) *If $(a_y = \mathrm{R})$ or $(a_x = \mathrm{F}$ and $b_x = 0)$, then $\gamma_{\mathrm{P},L}(G) = \gamma_{\mathrm{P},L'}(G')$.*

(3) *If $a_x = \mathrm{B}$ and $b_y > 0$, then $\gamma_{\mathrm{P},L}(G) = \gamma_{\mathrm{P},L'}(G')$, where b'_y is redefined to be $b_y - 1$.*

(4) *If $a_x = \mathrm{B}$ and $b_y = 0$, then $\gamma_{\mathrm{P},L}(G) = \gamma_{\mathrm{P},L'}(G')$, where a'_y is redefined to be* R.

(5) *Otherwise, if $a_x = \mathrm{F}$, $b_x > 0$ and $a_y \neq \mathrm{R}$, then $\gamma_{\mathrm{P},L}(G) = \gamma_{\mathrm{P},L'}(G')$, where a'_y is redefined to be* F.

An example of application of these rules for 2-power domination is depicted in Figure 1: the labels obtained from deleting the leaves bottom-up, from left to right are given.

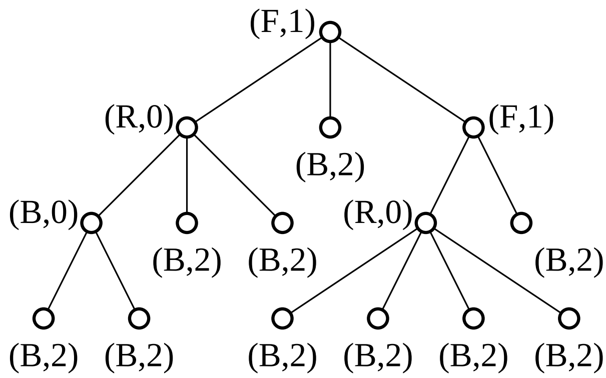

Figure 1: Example of labels after application of the algorithm.

Proof: (1) Suppose S is a $\gamma_{\mathrm{P},L}(G)$-set. Since the label a_x is R, x is necessarily in S. Let $S' = S \setminus \{x\}$. We prove by induction on i that for any step i,

$\mathcal{P}^i_{G',L'}(S') = \mathcal{P}^i_{G,L}(S) \setminus \{x\}$. Clearly, since a'_y was redefined not to be B, this is true initially, for $i = 0$. Suppose the property is true until some step $i \geq 0$. Let $v \in \mathcal{P}^i_{G,L}(S) \setminus \{x\}$. By induction hypothesis, $v \in \mathcal{P}^i_{G',L'}(S')$ and $N_G[v] \setminus \mathcal{P}^i_{G,L}(S) = N_{G'}[v] \setminus \mathcal{P}^i_{G',L'}(S')$. Added to the fact that the b_v are unchanged, we obtain that if v satisfies the L-PC in G, it also satisfies the L'-PC in G', and vice versa. Thus, $\mathcal{P}^i_{G',L'}(S') = \mathcal{P}^i_{G,L}(S) \setminus \{x\}$ for all i and so S' is an L'-power dominating set of G'. This gives $\gamma_{\mathrm{P},L}(G) \geq \gamma_{\mathrm{P},L'}(G') + 1$.

On the other hand, suppose S' is a $\gamma_{\mathrm{P},L'}(G')$-set. Let $S = S' \cup \{x\}$. Similarly, $\mathcal{P}^i_{G,L}(S) = \mathcal{P}^i_{G',L'}(S') \cup \{x\}$ for all i and so S is an L-power dominating set of G. This gives $\gamma_{\mathrm{P},L}(G) \leq \gamma_{\mathrm{P},L'}(G') + 1$.

In the following, $a_x \neq$ R. Suppose S is a $\gamma_{\mathrm{P},L}(G)$-set. We may assume that $x \notin S$, for otherwise the set $S^\dagger = S \setminus \{x\} \cup \{y\}$ satisfies that $\mathcal{P}^0_{G,L}(S) \subseteq \mathcal{P}^0_{G,L}(S^\dagger)$ and so $S^\dagger$ is a $\gamma_{\mathrm{P},L}(G)$-set which can be used to replace S.

(2) First remark that if $a_y =$ R, then the property clearly holds. Suppose now that $a_x =$ F and $b_x = 0$.

Consider a subset of vertices S not containing x. We claim that $\mathcal{P}^i_{G',L'}(S) \cup \{x\} = \mathcal{P}^i_{G,L}(S)$ for all i. Clearly, this is true for $i = 0$. Suppose it is true until some step $i \geq 0$. We check that any vertex satisfying the L'-PC in G' at step i also satisfies the L-PC in G, and vice versa. The only vertex that may satisfy the L'-PC in G' but not the L-PC in G is y. But since $a_x = F$, $x \in \mathcal{P}^0_{G,L}(S) \subseteq \mathcal{P}^i_{G,L}(S)$, and therefore $N_G[y] \setminus \mathcal{P}^i_{G,L}(S) \subseteq N_{G'}[y] \setminus \mathcal{P}^i_{G',L'}(S)$. By induction hypothesis, any vertex (different from x) satisfying the L-PC in G satisfies the L'-PC in G'. In the case of x, since $b_x = 0$, the satisfaction of the L-PC is not relevant for the computation of the monitored set at the next step.

(3) Consider a $\gamma_{\mathrm{P},L}(G)$-set S not containing x. We claim that S is a L'-power dominating set of G'. To prove it, we show that for any $i \geq 0$, $\mathcal{P}^i_{G,L}(S) \setminus \{x\} \subseteq \mathcal{P}^i_{G',L'}(S)$. Clearly, this is true when $i = 0$. Suppose it is true until step i. Let $v \neq x$ satisfy the L-PC condition in G. If $v \neq y$, then $b_v = b'_v$ and v satisfies the L'-PC in G' by induction hypothesis. Suppose now that y satisfies the L-PC.

First assume that $x \in \mathcal{P}^i_{G,L}(S)$. Then, either $x \in \mathcal{P}^0_{G,L}(S)$, which implies that $y \in S$ and $N[y] \subseteq \mathcal{P}^0_{G,L}(S)$, or x was added later, meaning that y satisfied the L-PC in G at some earlier step $i^* < i$. In both cases, $N_G[y] \subseteq \mathcal{P}^i_{G,L}(S)$ and by induction hypothesis, $N_{G'}[y] \subseteq \mathcal{P}^i_{G',L'}(S')$, thus y also satisfies the L'-PC condition in G' ($b'_y \geq 0$).

Suppose now that $x \notin \mathcal{P}^i_{G,L}(S)$. Then $N_G[y] \setminus \mathcal{P}^i_{G,L}(S) = N_{G'}[y] \setminus \mathcal{P}^i_{G',L'}(S) \dot\cup \{x\}$. Therefore, the hypothesis that $\left|N_G[y] \setminus \mathcal{P}^i_{G,L}(S)\right| \leq b_y$ implies that $\left|N_{G'}[y] \setminus \mathcal{P}^i_{G',L'}(S)\right| \leq b_y - 1 = b'_y$, and y satisfies the L'-PC in G'.

Finally, remark that if x satisfies the L-PC in G, then $x \in \mathcal{P}^i_{G,L}(S)$ and as remarked above, y is also in $\mathcal{P}^i_{G,L}(S)$. Therefore, this is not relevant for the computation of the monitored set at the next step. This concludes the proof of our claim.

Now, consider a $\gamma_{\mathrm{P},L'}(G')$-set S'. First remark that for any step $i \geq 0$, $\mathcal{P}^i_{G',L'}(S') \subseteq \mathcal{P}^i_{G,L}(S')$. Indeed, the only reason for this assertion not to be true would be that at some step i, y satisfies the L'-PC in G' but not the L-PC in G. However, since $N_G[y] \setminus N_{G'}[y] = \{x\}$ and $b_y = b'_y + 1$, this may not happen. Moreover, since S' is a L'-power dominating set of G', y has to satisfy the L'-PC in G' at some step, thus the L-PC in G, and x gets monitored at the next step.

(4) Any L'-power dominating set S' of G' contains y since $a'_y = \mathrm{R}$, so S' is also an L-power dominating set of G.

From our earlier remark, we may consider a $\gamma_{\mathrm{P},L}(G)$-set S that does not contain x. Since $b_y = 0$, y may not satisfy the L-PC unless x is in $\mathcal{P}^0_{G,L}(S)$. Yet, $a_x = \mathrm{B}$, so for x to be in $\mathcal{P}^0_{G,L}(S)$, it need to be in $N[S]$. Therefore, y is in S, and S is a L'-power dominating set of G'.

(5) Suppose now that $a_x = \mathrm{F}$, $b_x > 0$ and $b_y \neq \mathrm{R}$. For any set S not containing x and for any step $i \geq 0$, we claim that $\mathcal{P}^i_{G,L}(S) \subseteq \mathcal{P}^i_{G',L'}(S) \cup \{x\} \subseteq \mathcal{P}^{i+1}_{G,L}(S)$, thus implying that S is a L-power dominating set of G if and only if S is a L'-power dominating set of G'.

Since $a'_y = \mathrm{F}$, the only neighbour of x (namely y) is in $\mathcal{P}^0_{G,L}(S)$ and the first inclusion is easy to check.

For the second inclusion, first remark that $\mathcal{P}^0_{G',L'}(S) \setminus \{y\} \subseteq \mathcal{P}^0_{G,L}(S)$. Moreover, x is in $\mathcal{P}^0_{G,L}(S)$ and satisfies the L-PC in G, so $y \in \mathcal{P}^1_{G',L'}(S)$. Thus, the inclusion is true for $i = 0$. Now, one can check that every vertex satisfying the L'-PC in G' at step i also satisfies the L-PC in G at step $i+1$. This concludes the proof. $\square$

3. Bounds for k-power domination

Let G be a connected graph. We first establish that the natural relation on the k-power domination numbers of a graph when k varies is best possible.

We also prove that if $n = |V(G)| \geq k+2$, then $\gamma_{P,k}(G) \leq \frac{n}{k+2}$. A better bound is also proposed when G is claw-free and $(k+2)$-regular.

3.1. Relation between k- and $(k+1)$-power domination

Comparing k-power domination for different values of k, we note that if S is a k-power dominating set, then it is also a k'-power dominating set for any larger k'. Thus, we clearly get for any graph G the following inequality chain:

$$\gamma(G) = \gamma_{P,0}(G) \geq \gamma_{P,1}(G) \geq \ldots \geq \gamma_{P,k-1}(G) \geq \gamma_{P,k}(G) \geq \gamma_{P,k+1}(G) \geq \ldots.$$

Actually, this inequality chain cannot be improved in general, as shows the following observation:

Observation 6. *If $(x_k)_{0\leq k\leq n}$ is a finite non-increasing sequence of positive integers, then there exists a graph G such that $\gamma_{P,k}(G) = x_k$ for $0 \leq k \leq n$.*

Proof: For $0 \leq k \leq n$, take $x_k - x_{k+1}$ copies of the star $K_{1,k+1}$, where x_{n+1} is set as 0, and form a complete subgraph on the centres of all these stars. An example of such a graph for the sequence $(7,5,5,3,2)$ is depicted by Figure 2. One can check easily that for a given k, a k-power dominating set must contain one vertex from each of the x_k stars $K_{1,j}$ where $j \geq k+1$, and that the centres of these stars form a $\gamma_{P,k}(G)$-set. $\square$

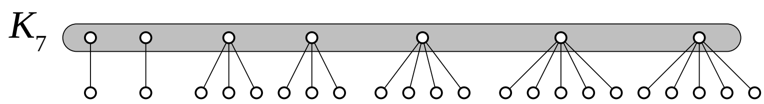

Figure 2: The graph for the k-power domination number sequence $(7,5,5,3,2)$.

3.2. General bounds

In this part, we mainly prove Theorem 11, which states that any graph G of order n satisfies $\gamma_{P,k}(G) \leq \frac{n}{k+2}$. We first need to prove the following few lemmas.

Lemma 7. *If G is connected and $\Delta(G) \leq k+1$, then $\gamma_{P,k}(G) = 1$.*

Proof: For any vertex v, we claim that $S = \{v\}$ is a k-power dominating set of G. Let $w \in \mathcal{P}^i(S)$, $i \geq 0$. At least one neighbour of w is in $\mathcal{P}^i(S)$, so $|N(w) \setminus \mathcal{P}^i(S)| \leq d(w) - 1 \leq k$ and then $N(w) \subseteq \mathcal{P}^{i+1}(S)$. This being true for any vertex w and for any $i \geq 0$, $\mathcal{P}^{i+1}(S) = N[\mathcal{P}^i(S)]$. By connectivity of the graph, $\mathcal{P}^\infty(S) = V(G)$. □

In the following we consider graphs with maximum degree at least $k + 2$.

Lemma 8. *If S is a k-power dominating set of G containing a vertex v of degree at most $k + 1$, then $(S \setminus \{v\}) \cup \{u\}$ is also a k-power dominating set of G for any $u \in N(v)$.*

Proof: Let $S' = (S \setminus \{v\}) \cup \{u\}$. The set $\mathcal{P}^0(S')$ contains both u and v. Therefore, $\left|N(v) \setminus \mathcal{P}^0(S')\right| \leq d(v) - 1 \leq k$, and $N[v] \subseteq \mathcal{P}^1(S')$. Thus, $\mathcal{P}^0(S) \subseteq \mathcal{P}^1(S')$ and by Observation 3, S' is a k-power dominating set of G. □

Lemma 9. *If G is a connected graph with maximum degree at least $k + 2$, then there exists a $\gamma_{\mathrm{P},k}(G)$-set containing only vertices of degree at least $k+2$.*

Proof: The proof is based on Lemma 8. Let S be a $\gamma_{\mathrm{P},k}(G)$-set containing as many vertices of degree at least $k+2$ as possible. Suppose S contains a vertex v of degree at most $k + 1$. Let w be a vertex of degree at least $k + 2$ closest to v, that is such that $j = d(v, w)$ is minimum. Consider a shortest path $\mu = (x_0, x_1, \ldots, x_j)$ from $x_0 = v$ to $x_j = w$. Iteratively applying Lemma 8, we obtain that for any $1 \leq i \leq j$, $S_i = (S \setminus \{v\}) \cup \{x_i\}$ is also a k-power dominating set of G. In particular, this is true for $x_j = w$, thus contradicting the assumption that S contains the maximum number of degree at least $k+2$ vertices, or the minimality of S. □

For any vertex v of a subset S of $V(G)$, the *S-private neighbourhood* of v, denoted $\mathrm{epn}(v, S)$, is the set of neighbours of v which are not neighbours of any vertex of $S \setminus \{v\}$.

Lemma 10. *If G is a connected graph with $\Delta(G) \geq k + 2$, then there exists a $\gamma_{\mathrm{P},k}(G)$-set S such that every vertex in S has at least $k + 1$ S-private neighbours, i.e., $|\mathrm{epn}(x, S)| \geq k + 1$ for $x \in S$.*

Proof: Let S be a $\gamma_{\mathrm{P},k}(G)$-set having only vertices of degree at least $k + 2$ (by Lemma 9) such that $G[S]$ has the minimum number of components. If every vertex x of S has $|\mathrm{epn}(x, S)| \geq k+1$, then we are done. Suppose there exists v in S such that $|\mathrm{epn}(v, S)| \leq k$. We consider two cases:

Case 1. There exists $w \in S$ adjacent to v. We claim that $S' = S \setminus \{v\}$ is a k-power dominating set of G, contradicting the minimality of S. Indeed, since $v \in \mathcal{P}^0(S')$ and $|N(v) \setminus \mathcal{P}^0(S')| = |\mathrm{epn}(v, S)| \leq k$, $N[v] \subseteq \mathcal{P}^1(S')$. Thus, $\mathcal{P}^0(S) \subseteq \mathcal{P}^1(S')$ and S' is a k-power dominating set of G, contradicting the minimality of S.

Case 2. Vertex v is an isolated vertex of $G[S]$. Since $d(v) \geq k+2$ and $|\mathrm{epn}(v, S)| \leq k$, there exist $w \in N(v)$ and $x \in S$ such that $w \in N(x)$ (w is not a S-private neighbour of x). Set $S' = (S \setminus \{v\}) \cup \{w\}$ if $d(w) \geq k+2$ and $S' = S \setminus \{v\}$ otherwise. We claim that S' is a k-power dominating set of G. Indeed, since $v \in \mathcal{P}^0(S')$ and $|N(v) \setminus \mathcal{P}^0(S')| \subseteq |\mathrm{epn}(v, S)| \leq k$, $N[v] \subseteq \mathcal{P}^1(S')$. Thus, $\mathcal{P}^0(S) \subseteq \mathcal{P}^1(S')$ and S' is a k-power dominating set of G. Moreover, S' has one less component than S, contradicting the choice of S. □

As a consequence of Lemmas 7 and 10, we obtain the following theorem.

Theorem 11. *If G is a connected graph of order $n \geq k+2$, then*

$$\gamma_{\mathrm{P},k}(G) \leq \frac{n}{k+2}$$

and this bound is best possible.

Proof: That $\frac{n}{k+2}$ is a upper bound for the k-power domination number of G is a direct consequence of Lemmas 7 and 10.

Let us describe graphs for which this bound is attained. Let G be any connected graph on n vertices, denoted $v_1, v_2, \ldots, v_n$ and let $H_1, H_2, \ldots, H_n$ be a family of graphs on $k+1$ vertices. We form a new graph G' by taking the disjoint union of all these graphs and adding edges linking v_i to every vertices of H_i.

We prove by contradiction that a k-power dominating set of such a graph G' must contain at least one vertex in each $V(H_i) \cup \{v_i\}$. Suppose there exists some k-power dominating set S not containing any vertex in $V(H_1) \cup \{v_1\}$, without loss of generality. At the beginning, no vertex of H_1 is monitored, $\mathcal{P}^0(S) \cap V(H_1) = \emptyset$. Let i be the smallest integer such that there exists $x \in V(H_1) \cap \mathcal{P}^i(S)$. It exists since S is a k-power dominating set, and $i \geq 1$. For x to be added into $\mathcal{P}^i(S)$, it means that some vertex from $N(x)$ was in $\mathcal{P}^{i-1}(S)$ and had at most k neighbours not in $\mathcal{P}^{i-1}(S)$. Since

$N(x) \subseteq V(H_1) \cup \{v_1\}$ and $V(H_1) \cap \mathcal{P}^{i-1}(S) = \emptyset$, this vertex is necessarily v_1. Yet $N[v_1] \setminus \mathcal{P}^{i-1}(S) \supseteq V(H_1)$ which contain $k+1$ vertices, contradicting our assumption.

Therefore, a k-power dominating set of G' must contain at least one vertex in each $V(H_i) \cup \{v_i\}$, and this is at least n vertices among the $n(k+2)$ vertices of G'. Note that $V(G)$ for example is a k-power dominating set of G'. $\square$

3.3. Regular claw-free graphs

This subsection is dedicated to *claw-free* graphs. A *claw-free* graph is a graph that does not contain a claw, i.e. $K_{1,3}$, as an induced subgraph.

For positive integers k and r, let $D_{k,r}$ be the graph obtained from the disjoint union of r copies of $D_i \cong K_{k+3} - x_i y_i$, a complete graph on $k+3$ vertices minus one edge $x_i y_i$, for $1 \le i \le r$ by adding r edges $y_i x_{i+1}$ ($1 \le i \le r$) where $x_{r+1} = x_1$; see Figure 3 for $D_{k,6}$.

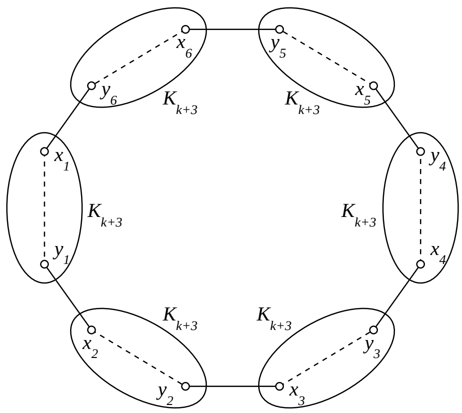

Figure 3: The graph $D_{k,6}$.

Theorem 12. *For a positive integer k, if G is a connected claw-free $(k+2)$-regular graph of n vertices, then $\gamma_{\mathrm{P},k}(G) \le n/(k+3)$ with equality if and only if G is isomorphic to $D_{k,r}$ for some $r \ge 1$.*

Proof: Let S be a $\gamma_{\mathrm{P},k}(G)$-set such that (1) $G[S]$ has as few edges as possible and under this condition, (2) $|N[S]|$ is as large as possible.

Claim 1. *$G[S]$ is an independent set.*

Proof: Suppose to the contrary that there exist two adjacent vertices in $G[S]$, say u and v. Observe first that $|\text{epn}(v,S)| = k+1$; otherwise, we have $|\text{epn}(v,S)| \leq k$ and $S' = S \setminus \{v\}$ is a k-power dominating set with less vertices than S, contradicting the minimality of S. Now let w be any S-private neighbour of v and consider $S'' = (S \setminus \{v\}) \cup \{w\}$. The set S'' is a $\gamma_{\text{P},k}(G)$-set, but $G[S'']$ contains less edges than $G[S]$, contradicting (1). This completes the proof of Claim 1. □

Claim 2. *If u and v are two distinct vertices of S, then $|N(u) \cap N(v)| \leq 1$.*

Proof: Suppose to the contrary that $|N(u) \cap N(v)| \geq 2$. Let x and y be two distinct vertices in $N(u) \cap N(v)$. As the graph is claw-free, G contains either the edge xy or the edge xz or yz where z is a third neighbour of v. In the two cases, $S' = S \setminus \{u\}$ is a k-power dominating set of G with one vertex less than S, contradicting the minimality of S. This completes the proof of Claim 2. □

Claim 3. *Let u and v be two distinct vertices of S and x be a vertex in $N(u) \cap N(v)$. Moreover assume that u and x are adjacent to a vertex y. Then, the $k+1$ other neighbours $z_1, z_2, \ldots, z_{k+1}$ of v are S-private neighbours of v, form a clique of size $k+1$, and $N[z_i] \subseteq N[S]$ for $1 \leq i \leq k+1$, see Figure 4.*

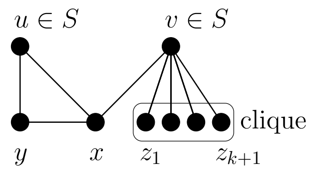

Figure 4: Claim 3.

Proof: First, observe that there is no edge xz_i; otherwise $S' = S \setminus \{v\}$ is a k-power dominating set with less vertices than S, contradicting the minimality of S. Second, since the graph is claw-free, it follows that the vertices z_i's form a clique of size $k+1$. If some z_i belongs to $N[S \setminus \{v\}]$, then as previously, $S' = S \setminus \{v\}$ is a k-power dominating set with less vertices than S, contradicting the minimality of S. To complete the proof, assume that $N[z_i] \not\subseteq N[S]$ for some i. Then $S'' = S \setminus \{v\} \cup \{z_i\}$ is a $\gamma_{\text{P},k}(G)$-set with $|N[S'']| > |N[S]|$, contradicting (2). This completes the proof of Claim 3. □

Claim 4. *Let v be a vertex of S with neighbourhood $x, z_1, z_2, \ldots, z_{k+1}$ such that x is adjacent to $u \in S \setminus \{v\}$ and to y with $uy \in E(G)$, and such that the vertices $z_1, z_2, \ldots, z_{k+1}$ form a clique of size $k+1$. Moreover assume that z_1 is linked to a vertex t which is linked to a vertex $w \in S \setminus \{v\}$. Then, t is also linked to the vertices $z_2, z_3, \ldots, z_{k+1}$. The $k+1$ other neighbours $s_1, s_2, \ldots, s_{k+1}$ of w are S-private neighbours of w, form a clique of size $k+1$, and $N[s_i] \subseteq N[S]$ for $1 \leq i \leq k+1$, see Figure 5.*

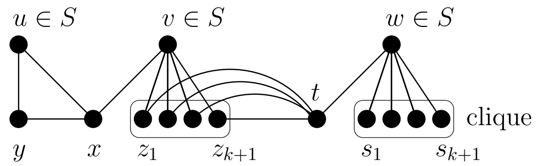

Figure 5: Claim 4.

Proof: First observe that there is no edge ts_i; otherwise $S' = S \setminus \{v\}$ is a k-power dominating set with less vertices than S, contradicting the minimality of S. It follows that, since the graph is claw-free, t is linked to all z_i ($1 \leq i \leq k+1$), and that the vertices $s_1, s_2, \ldots, s_{k+1}$ form a clique of size $k+1$. Now, every s_i is a S-private neighbour of w; otherwise, $S'' = S \setminus \{w\}$ is a k-power dominating set with less vertices than S, contradicting the minimality of S. Finally, $N[s_i] \subseteq N[S]$ for all i; otherwise, $S''' = S \setminus \{v, w\} \cup \{z_1, s_i\}$ is a $\gamma_{\mathrm{P},k}(G)$-set with $|N[S''']| > |N[S]|$, contradicting (2). This completes the proof of Claim 4. □

By using similar arguments, one can extend the previous claim as follows:

Claim 5. *Let $u, v_1, v_2, \ldots, v_i$ be distinct vertices of S. Every v_j with $1 \leq j \leq i$ has neighbourhood $t_{i-1}, z_1^j, z_2^j, \ldots, z_{k+1}^j$. Assume for all $1 \leq j \leq i$, the vertices $z_1^j, z_2^j, \ldots, z_{k+1}^j$ form a clique of size $k+1$ and are linked to t_j. The vertex u is linked to two vertices y and t_0, and $yt_0 \in E(G)$. Finally suppose that z_1^i is linked to a vertex t which is linked to a vertex $v_{i+1} \in S \setminus \{v_1, v_2, \ldots, v_i\}$. Then, t is also linked to the vertices $z_2^i, z_3^i, \ldots, z_{k+1}^i$. The $k+1$ other neighbours $z_1^{i+1}, z_2^{i+1}, \ldots, z_{k+1}^{i+1}$ of v_{i+1} are S-private neighbours of v_{i+1}, form a clique of size $k+1$, and $N[z_j^{i+1}] \subseteq N[S]$ for $1 \leq j \leq k+1$, see Figure 6.*

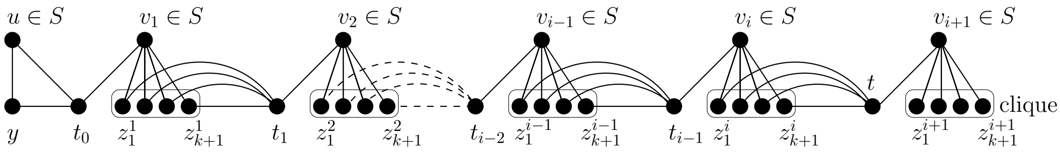

Figure 6: Claim 5.

Claim 6. *If u and v are two distinct vertices of S, then $N(u) \cap N(v) = \emptyset$.*

Proof: Assume that two vertices u and v of S have a common neighbour x. Since x must not be in a claw and by symmetry, there exists a vertex y linked to u and x. We apply now Claims 3, 4 and repetitively Claim 5. Since S is finite, we obtain with Claim 5 that $v_{i+1} = u$ and $z_j^{i+1} = t_0$ for some j which is not a S-private neighbour of v_{i+1}, a contradiction. □

It follows that $|V(G)| \geq \gamma_{\mathrm{P},k}(G)(k+3)$ and so $\gamma_{\mathrm{P},k}(G) \leq |V(G)|/(k+3)$.

Now observe that $\gamma_{\mathrm{P},k}(D_{k,r}) = r = |V(D_{k,r})|/(k+3)$. Suppose that $\gamma_{\mathrm{P},k}(G) = |V(G)|/(k+3)$. By Claim 6, one can choose S such that the $N[v]$'s with $v \in S$ are a partition of $V(G)$. A vertex $z \in N(S)$ is said to be *special* if there are two distinct vertices u and v in S such that z is linked to v and $N(z) \subseteq N(u) \cup \{v\}$.

Claim 7. *Let u and v be two distinct vertices of S. Assume that u is adjacent to two adjacent vertices x and y. Finally, let z be a vertex adjacent to x and v. Then, z is special. Moreover $G[N(v) \setminus \{z\}]$ is a clique of size $k+1$ and so it is for $G[N(z) \cap N(u)]$.*

Proof: Suppose that z is not special. Then, $N(z)$ contains at most k vertices of $N(u)$. Let r be a vertex in $N(z) \setminus (N(u) \cup \{v\})$. By the choice of S, r belongs to $N(S)$ (like every vertex not in S). It follows that $S' = S \setminus \{u\}$ is a k-power dominating set having less vertices than S, contradicting the minimality of S. Hence, z is special. Since the graph is claw-free and $N(u) \cap N(v) = \emptyset$, $G[N(z) \cap N(u)]$ is a clique of size $k+1$. Finally, the graph being claw-free, v, $G[N(v) \setminus \{z\}]$ is also a clique of size $k+1$. □

If $|S| = 1$, then $G = K_{k+3} = D_{k,1}$. So assume that $|S| \geq 2$. Let $A(v) = \{x \in N(v) \colon N(x) \setminus N[v] \neq \emptyset\}$. First observe that for all $v \in S$, $|A(v)| \geq 2$. If every $v \in S$ has $|A(v)| = 2$, then by connectedness, G is $D_{k,|S|}$

as desired. Let v_1 be a vertex of S with $|A(v_1)| \geq 3$. Since the graph is claw-free v_1 is adjacent to two adjacent vertices x_1 and y_1 with $x_1, y_1 \in A(v_1)$. Moreover we can assume that x_1 is linked to a vertex z_2, which is linked to a vertex $v_2 \in S \setminus \{v_1\}$. By Claim 7, z_2 is special, and both $G[N(z_2) \cap N(v_1)]$ and $G[N(v_2) \setminus \{z_2\}]$ are cliques of size $k+1$. Now, there exist two adjacent vertices x_2, y_2 in $N(v_2) \setminus \{z_2\}$ with $x_2 \in A(v_2)$. By repeating the same argument, we have $S = \{v_1, v_2, \ldots, v_r\}$; each $N(v_i)$ contains a special vertex z_i such that $G[N(v_i) \setminus \{z_i\}]$ is a clique of size $k+1$ and $N(z_i) = \{v_i\} \cup N(v_{i-1}) \setminus \{z_{i-1}\}$ for $1 \leq i \leq r$, where $v_0 = v_r$ and $z_0 = z_r$. This implies that $G = D_{k,r}$. This completes the proof of Theorem 12. □

3.4. Graphs of diameter two

In [16], the authors propose some bounds on the power domination number of planar graphs when the diameter is bounded. A natural question is then whether one can propose a general (constant) bound on the k-power domination number of graphs with bounded diameter. The answer to this question is negative, as the following proposition shows.

Proposition 13. *For any k, there exist graphs of diameter 2 with arbitrarily large k-power domination number.*

Proof: Suppose (P, L) is a finite projective plane of order n, where the set P contains exactly $n^2 + n + 1$ points, the set L contains exactly $n^2 + n + 1$ lines, each point is in exactly $n + 1$ lines and each line contains exactly $n + 1$ points. Consider the graph G with vertex set $V(G) = P \cup L$ and edge set $E(G) = \{x\ell : x \in P, \ell \in L, x \in \ell\} \cup \{\ell\ell' : \ell \neq \ell' \text{ in } L\}$.

Since every two points are in exactly one line, G is of diameter 2.

If S is a $\gamma_{\mathrm{P},k}(G)$-set, then without loss of generality we may assume that $S \subseteq L$. Since $N[S] = L \cup \bigcup\{x \in \ell : \ell \in S\}$, any line not in S is adjacent to $n + 1 - |S|$ points which are not in $N[S]$. In order for S to be a k-power dominating set of G, it must be the case that $n + 1 - |S| \leq k$ or equivalently $|S| \geq n + 1 - k$. This gives that $\gamma_{\mathrm{P},k}(G) \geq n + 1 - k$.

On the other hand, consider a point x and a subset $S \subseteq L$ of size $n+1-k$, such that any line in S contains x. The set $N[S]$ contains all lines. Moreover, any line ℓ that do not contain x share a point with every line in S, and these points are all distinct. Indeed, if two lines in S shared the same point with ℓ, then they would share both x and that point, but by any two points goes only one line. Therefore, any line not containing x contain $n-k+1$ points in $N[S]$.

Hence, $\{N[\ell], x \notin \ell\} \subseteq \mathcal{P}^1(S)$, and S k-power dominates G. Consequently, $\gamma_{\mathrm{P},k}(G) \leq n+1-k$ and so $\gamma_{\mathrm{P},k}(G) = n+1-k$. □

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