Steinberg’s Conjecture and near-colorings

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Abstract

Let $\mathcal{F}$ be the family of planar graphs without cycles of length 4 and 5. Steinberg’s Conjecture (1976) that says every graph of $\mathcal{F}$ is 3-colorable remains widely open. Focusing on a relaxation proposed by Erdős (1991), many studies proved the conjecture for some subfamilies of $\mathcal{F}$. For example, Borodin et al. proved that every planar graph without cycles of length 4 to 7 is 3-colorable. In this note we propose to relax the problem not on the family of graphs but on the coloring by considering near-colorings. A graph $G = (V, E)$ is said to be $(i, j, k)$-colorable if its vertex set can be partitioned into three sets $V_1, V_2, V_3$ such that the graphs $G[V_1], G[V_2], G[V_3]$ induced by the sets $V_1, V_2, V_3$ have maximum degree at most $i, j, k$ respectively. Under this terminology, Steinberg’s Conjecture says that every graph of $\mathcal{F}$ is $(0, 0, 0)$-colorable. A result of Xu (2008) implies that every graph of $\mathcal{F}$ is $(1, 1, 1)$-colorable. Here we prove that every graph of $\mathcal{F}$ is $(2, 1, 0)$-colorable and $(4, 0, 0)$-colorable.

1 Introduction

In 1976, Appel and Haken proved that every planar graph is 4-colorable [2, 3], and as early as 1959, Grötzsch [20] showed that every planar graph without 3-cycles is 3-colorable. As proved by Garey, Johnson and Stockmeyer [19], the problem of deciding whether a planar graph is 3-colorable is NP-complete. Therefore, some sufficient conditions for planar graphs to be 3-colorable were stated. In 1976, Steinberg [24] raised the following:

Steinberg’s Conjecture ’76 Every planar graph without 4- and 5-cycles is 3-colorable.

There were then no progress in this direction until Erdős (1991) proposed the following relaxation of Steinberg’s Conjecture:

Erdős’ relaxation ’91 Determine the smallest value of $k$, if it exists, such that every planar graph without cycles of length from 4 to $k$ is 3-colorable.

Abbott and Zhou [1] proved that such a $k$ does exist, with $k \leq 11$. This result was later on improved to $k \leq 10$ by Borodin [4], to $k \leq 9$ by Borodin [5] and Sanders and Zhao [22], to $k \leq 8$

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by Salavatipour [21]. The best known bound for such a $k$ is 7 which was proved by Borodin, Glebov, Raspaud and Salavatipour [10].

This approach was at the origin of sufficient conditions of 3-colorability of subfamilies of planar graphs where some families of cycles are forbidden. See for examples [8, 9, 12, 13, 14, 15, 16, 17, 25].

A graph $G$ is called improperly $(d_1, d_2, \ldots, d_k)$-colorable, or simply $(d_1, d_2, \ldots, d_k)$-colorable, if the vertex set of $G$ can be partitioned into subsets $V_1, V_2, \ldots, V_k$ such that the graph $G[V_i]$ induced by $V_i$ has maximum degree at most $d_i$ for $1 \leq i \leq k$. This notion generalizes those of proper $k$-coloring (when $d_1 = d_2 = \ldots = d_k = 0$) and $d$-improper $k$-coloring (when $d_1 = d_2 = \ldots = d_k = d \geq 0$). Under this terminology, the Four Color Theorem says that every planar graph is $(0, 0, 0, 0)$-colorable.

Eaton and Hull [18] and independently Škrekovski [23] proved that every planar graph is 2-improperly 3-colorable (in fact, 2-improperly 3-choosable), i.e. $(2, 2, 2)$-colorable.

In this note we focus on near-colorings and Steinberg’s Conjecture. Let $\mathcal{F}$ be the family of planar graphs without cycles of length 4 and 5. We prove:

**Theorem 1** Every graph of $\mathcal{F}$ is $(2, 1, 0)$-colorable and $(4, 0, 0)$-colorable.

The remaining of the paper is dedicated to the proof of this theorem.

## 2 General setting for $(s_1, s_2, s_3)$-colorability of $\mathcal{F}$

The proof of the main theorem is done by reducible configurations and discharging procedure. Suppose the theorem is not true. Let $G = (V, E, F)$ be a counterexample with the minimum order embedded in the plane. We apply a discharging procedure to reach to a contradiction.

We first assign to each vertex $v$ and face $f$ of $G$ a charge $\omega$ such that $\omega(v) = 2d(v) - 6$ and $\omega(f) = r(f) - 6$, where $d(v)$ and $r(f)$ denote the degree of the vertex $v$ and the length of the face $f$ respectively. By Euler’s Formula $|V| - |E| + |F| = 2$ and formula $\sum_{v \in V} d(v) = 2|E| = \sum_{f \in F} r(f)$, we have:

$$\sum_{v \in V} \omega(v) + \sum_{f \in F} \omega(f) = -12 < 0.$$ 

We then redistribute the charges according to some discharging rules. During the process, no charges are created or disappear. Let $\omega^*$ be the new charge on each vertex and face after the procedure. It follows that:

$$\sum_{v \in V} \omega(v) + \sum_{f \in F} \omega(f) = \sum_{v \in V} \omega^*(v) + \sum_{f \in F} \omega^*(f).$$

However, we will show that under some structural properties of $G$ the new charge on each vertex and face is non-negative. This leads to the following obvious contradiction

$$-12 = \sum_{v \in V} \omega(v) + \sum_{f \in F} \omega(f) = \sum_{v \in V} \omega^*(v) + \sum_{f \in F} \omega^*(f) > 0$$

implying that no counterexample can exist.

Establishing structural properties is essential in the proof of the theorem. Although the properties for $(2, 1, 0)$-coloring and for $(4, 0, 0)$-coloring are not the same, they share some common part. In this section, we derive lemmas for a general setting. Suppose $s_1 \geq s_2 \geq s_3 \geq 0$ and $s = s_1 + s_2 + s_3$. In this section we assume that $G$ is a minimum counterexample in $\mathcal{F}$ that is not $(s_1, s_2, s_3)$-colorable.

A vertex of degree $k$ (resp. at least $k$, at most $k$) will be called $k$-vertex (resp. $k^+$-vertex, $k^-$-vertex). A similar notation will be used for cycles and faces. A $k$-neighbor (resp. $k^+$-neighbor, $k^-$-neighbor) of some vertex $u$ is a neighbor of $u$ which is a $k$-vertex. An $(a, b, c)$-face is a 3-face $uvw$ such that $d(u) = a$, $d(v) = b$ and $d(w) = c$. In addition, $a^-$ (resp. $a^+$) will mean $d(u) \leq a$
(resp. $d(u) \geq a$) and $\ast$ will mean any degree. For example, a $(3, 4^-, \ast)$-face is a 3-face $uvw$ such that $d(u) = 3$, $d(v) \leq 4$ and $w$ has no restriction on its degree. A pendent 3-face of a vertex $v$ is a 3-face not containing $v$ but is incident to a 3-vertex adjacent to $v$. In the following we will color the vertices of the graphs by partitioning the vertex set into $V_1, V_2, V_3$ such that each $V_i$ induces a subgraph of maximum degree at most $s_i$. Coloring a vertex with color $i$ means adding the vertex into $V_i$. We will say that we nicely color a vertex if we color it by $i$ and at most $\max\{0, s_i - 1\}$ of its neighbors are colored by $i$. We say that we properly color a vertex if we color it by a color not used by its neighbors. Properly colored vertices are nicely colored. When the colored neighbors of an uncolored vertex $v$ use at most two colors, in particular when $v$ has at most two colored neighbors, we can always color $v$ properly by using the third color not used by its neighbors. We will use this frequently. As an easy consequence, every vertex of $G$ has degree at least 3.

First, since $G$ has no 4-cycles, we have the following:

**Observation 2** Two 3-faces may not share an edge. If a $k$-vertex $v$ is incident to $\alpha$ 3-faces and has $\beta$ pendent 3-faces, then $2\alpha + \beta \leq k$.

Next, three useful lemmas.

**Lemma 3** Let $v$ be an $(s + 2)^-$-vertex of $G$. If $G - v$ has an $(s_1, s_2, s_3)$-coloring such that all neighbors of $v$ are nicely colored, then $G$ is $(s_1, s_2, s_3)$-colorable.

**Proof.** For $1 \leq i \leq 3$, if we cannot assign color $i$ to $v$, then $v$ has at least $s_i + 1$ neighbors colored by $i$. It follows that $v$ has degree at least $\sum_{i=1}^{3} (s_i + 1) = s + 3$, a contradiction. □

**Lemma 4** Graph $G$ contains no $(s + 2)^-$-vertex $v$ adjacent only to $4^-$-vertices, each 4-neighbor of which is adjacent some 3-neighbor of $v$.

**Proof.** Suppose to the contrary that $G$ contains such a $(s + 2)^-$-vertex $v$. By the minimality of $G$, the graph $G'$ obtained from $G$ by deleting $v$ and all of its neighbors admits an $(s_1, s_2, s_3)$-coloring. We first color all 4-neighbors of $v$ properly, and then color all 3-neighbors of $v$ properly. Then all neighbors of $v$ are nicely colored. Thus, by Lemma 3, $G$ is $(s_1, s_2, s_3)$-colorable, a contradiction. □

**Lemma 5** The three neighbors $x_1, x_2, x_3$ of a 3-vertex $v$ of $G$ use different colors in an $(s_1, s_2, s_3)$-coloring of $G - v$. Moreover, assume $x_1$ is colored by $i$, we have $d(x_i) \geq s_i + 3$ for $1 \leq i \leq 3$. Furthermore, if $s_i > 0$ and $x_i$ is adjacent to $x_j$, then either $d(x_i) > s_i + 3$ or $d(x_j) > s_j + 3$.

**Proof.** If $x_1, x_2, x_3$ do not use three distinct colors, then we can properly color $v$, a contradiction. Hence w.l.o.g. we can assume that $x_i$ is colored by $i$ for $1 \leq i \leq 3$.

Suppose for a contradiction that some $d(x_i) \leq s_i + 2$ for some $i$. Then $s_i + 1$ as $d(x_i) \geq 3$. If $x_i$ is nicely colored by $i$, then we color $v$ by $i$ and this extends the coloring to $G$, a contradiction. Hence, $x_i$ has at least $s_i$ neighbors colored by $i$. Since $x_i$ has an uncolored neighbor $v$, there is at least one color different from $i$ not used by its neighbors. We then color $v$ by $i$ and recolor $x_i$ by the unused color. This extends the coloring to $G$, a contradiction.

Suppose for a contradiction that $x_i$ is adjacent to $x_j$, but $d(x_i) = s_i + 3$ and $d(x_j) = s_j + 3$. Let $k$ be the color different from $i$ and $j$. Since $G$ has no 4-cycle, $x_k$ is not adjacent to $x_i$ and $x_j$. As above, $x_i$ (resp. $x_j$) has $s_i$ (resp. $s_j$) neighbors colored by $i$ (resp. $j$) and another colored neighbor $x_i'$ (resp. $x_j'$) other than $x_j$ (resp. $x_i$). If $x_i'$ is colored by $j$, then we may color $v$ by $i$ and recolor $x_i$ by $k$ to get an $(s_1, s_2, s_3)$-coloring of $G$, a contradiction. Hence, $x_i'$ is colored by $k$. Similarly, $x_j'$ is also colored by $k$. Then we may color $v$ by $i$, recolor $x_i$ by $j$ and recolor $x_j$ by $i$ to get an $(s_1, s_2, s_3)$-coloring of $G$ (notice that $s_i > 0$), again a contradiction. Hence, $d(x_i) > s_i + 3$ or $d(x_j) > s_j + 3$. □

**3** $(2, 1, 0)$-colorability of $\mathcal{F}$

In this section we prove that every graph in $\mathcal{F}$ is $(2, 1, 0)$-colorable, namely we consider the case $(s_1, s_2, s_3) = (2, 1, 0)$ for which $s = s_1 + s_2 + s_3 = 3$. 

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3.1 Reducible configurations for (2, 1, 0)-coloring

We first establish structural properties of $G$. More precisely, we prove that some 'configurations', i.e. subgraphs, are 'reducible', i.e. cannot appear in $G$ because it is a minimum counterexample. Lots of this configurations are depicted in Figure 1.

A light 5-vertex is a 5-vertex incident to a (3, 5, 5)-face $f$ and adjacent to three 3-vertices not in $f$. A poor (3, 5, 5)-face is a (3, 5, 5)-face incident to a light 5-vertex. If a 3-vertex is incident to a 3-face, then its neighbor not incident to this 3-face is said to be its outer neighbor.

As already mentioned we have the following.

(C1) $G$ contains no 2-face.

The two following claims come from Lemma 4 with $s = 3$.

(C2) $G$ contains no 5-vertex adjacent to five 3-vertices.

(C3) $G$ does not contain 5-vertices $v$ incident to a (3, 4, 5)-face $f$ and adjacent to three 3-vertices not in $f$.

(C4) $G$ contains no non-light 5-vertex incident to a poor (3, 5, 5)-face and a (3, 5, 5)-face, and adjacent to a 3-vertex not in these faces.

Proof. Suppose to the contrary that $G$ contains such a 5-vertex $v$. Let $uvw$ be the poor (3, 5, 5)-face, and the (3, 5, 5)-face with $d(u) = d(r) = 3$, and $x$ be the neighbor of $v$ not in these faces. Vertex $w$ is light and thus is adjacent to three 3-vertices distinct from $u$, say $w_1, w_2, w_3$. By the minimality of $G$, the graph $G - \{u, v, w, w_1, w_2, w_3, r, x\}$ admits a $(2, 1, 0)$-coloring. Now we extend this coloring as follows. We may assume that, if $s$ is colored by 1, then it has at most one neighbor colored by 1, otherwise we can properly recolor it. Then we color $r$ and $x$ properly. If $s, r, x$ use different colors, then we color $v$ with 1; otherwise we color $v$ properly. We then color $u, w_1, w_2, w_3$ properly. It follows that all neighbors of $w$ are nicely colored. By Lemma 3, $G$ is $(2, 1, 0)$-colorable, a contradiction. \qed

(C5) $G$ does not contain a poor (3, 5, 5)-face incident to two light 5-vertices.

Proof. Suppose to the contrary that $G$ contains a poor (3, 5, 5)-face $uvw$ with light vertices $v$ and $w$. For $x \in \{v, w\}$, let $x_1, x_2, x_3$ be the three neighbors of $x$ not in $\{u, v, w\}$. By the minimality of $G$, the graph $G - \{u, v, w, w_1, w_2, w_3, v_1, v_2, v_3\}$ admits a $(2, 1, 0)$-coloring. We extend the coloring to $\{v_1, v_2, v_3\}$ by coloring each of them properly. If $v_1, v_2, v_3$ use three distinct colors, then we color $v$ with 1, and properly otherwise. After this, we color $u, w_1, w_2, w_3$ properly. It follows that all neighbors of $w$ are nicely colored. By Lemma 3, $G$ is $(2, 1, 0)$-colorable, a contradiction. \qed

Let $v$ be a 3-vertex adjacent to three vertices $y_1, y_2, y_3$. Consider $G - v$. By Lemma 5, the colors 1, 2, and 3 appear on the neighbors of $v$. Moreover the vertex colored with 1 (resp. 2, 3) has degree at least 5 (resp. 4, 3). Thus (C6) and (C7) follow.

(C6) $G$ does not contain 3-vertices adjacent to two 3-vertices.

(C7) If $uvw$ is a (3, 4, 4)-face with $d(u) = 3$, then the outer neighbor of $u$ has degree at least 5.

Now, if the three vertices $y_1, y_2, y_3$ satisfy $d(y_1) = 3, d(y_2) \leq 4$ and $d(y_3) \leq d(y_3)$, then $y_1$ (resp. $y_2, y_3$) is colored with 3 (resp. 2, 1) and has degree 3 (resp. 4, at least 5). By the last sentence of Lemma 5, the vertices $y_1, y_2$ are non-adjacent; moreover if $d(y_3) = 5$, then $y_3$ is not adjacent to $y_1$ or $y_2$. Thus (C8), (C9), and (C10) follow.

(C8) $G$ does not contain $3, 3, 4^-$-faces.

(C9) If $uvw$ is a (3, 3, 5)-face with $d(u) = 3$, then the outer neighbor of $u$ has degree at least 5.

(C10) If $uvw$ is a $(3, 4, 5)$-face with $d(u) = 3, d(v) = 4$ and $d(w) = 5$, then the outer neighbor of $u$ has degree at least 4.
Figure 1: Reducible configurations (C2)-(C10). Black dots represent vertices all neighbours of which are drawn in the figure; the white dots represent vertices that can have nondepicted neighbours. Dashed lines represent edges that may possibly not exist.
3.2 Discharging procedure for \((2, 1, 0)\)-coloring

We now apply a discharging procedure to reach to a contradiction. The discharging rules are as follows:

- **R1.** Every 4-vertex gives \(\frac{1}{2}\) to each pendant 3-face.
- **R2.** Every 5\(^{+}\)-vertex gives 1 to each pendant 3-face.
- **R3.** Every 4-vertex gives 1 to each incident 3-face.
- **R4.** Every non-light 5-vertex gives 2 to each incident poor \((3, 5, 5)\)-face.
- **R5.** Every 5-vertex gives \(\frac{3}{2}\) to each incident non-poor \((3, 5, 5)\)-face or \((3, 4, 5)\)-face.
- **R6.** Every 5-vertex gives 1 to each other incident 3-face.
- **R7.** Every 6\(^{+}\)-vertex gives 2 to each incident 3-face.

Let \(v\) be a \(k\)-vertex with \(k \geq 3\) by (C1).

**Case** \(k = 3\). The discharging procedure does not involves 3-vertices. Hence \(\omega^*(v) = \omega(v) = 0\).

**Case** \(k = 4\). Initially \(\omega(v) = 2\). Vertex \(v\) gives 1 to each of the \(\alpha\) incident 3-faces by R3 and \(\frac{1}{2}\) to each of the \(\beta\) pendant 3-faces by R1. By Observation 2, \(\omega^*(v) = 2 - (\alpha + \frac{1}{2})\beta = 2 - \frac{1}{2} \cdot 4 = 0\).

**Case** \(k = 5\). Initially \(\omega(v) = 4\). Assume \(v\) is not incident to any 3-face. By (C2), \(v\) is adjacent to at most four 3-vertices and so has at most four pendant 3-faces. By R2, \(\omega^*(v) = 4 - 4 \cdot 1 = 0\).

Assume \(v\) is incident to exactly one 3-face \(f\). If \(v\) is a non-light 5-vertex and \(f\) is a poor \((3, 5, 5)\)-face, then \(v\) has at most two pendant 3-faces by definition. By R4 and R2, \(\omega^*(v) = 4 - 2 - 2 \cdot 1 = 0\).

If \(f\) is a non-poor \((3, 5, 5)\)-face, then \(v\) has at most two pendant 3-faces by definition. By R5 and R2, \(\omega^*(v) = 4 - \frac{3}{2} - 2 \cdot 1 > 0\). If \(f\) is a 3-face of other type, then by R6 and R2 \(\omega^*(v) = 4 - 1 - 3 \cdot 1 = 0\).

Assume \(v\) is incident to exactly two 3-faces \(f_1\) and \(f_2\). If \(v\) gives twice at most \(\frac{3}{2}\) to the 3-faces, then \(\omega^*(v) = 4 - 2 \cdot \frac{3}{2} - 1 = 0\). So we may assume that \(f_1\) or \(f_2\), say \(f_1\), is a poor \((3, 5, 5)\)-face. If \(f_2\) is a \((3, 5^-, 5)\)-face, then \(v\) has no pendant 3-faces by (C4) and \(\omega^*(v) = 4 - 2 - 2 = 0\). If \(f_2\) is a 3-face of other type, then \(v\) may have a pendant 3-face and \(\omega^*(v) = 4 - 2 - 1 - 1 = 0\) by R6.

**Case** \(k \geq 6\). Initially \(\omega(v) = 2k - 6\). Vertex \(v\) gives 2 to each of the \(\alpha\) incident 3-faces by R7 and 1 to each of the \(\beta\) pendant 3-faces by R2. By Observation 2, \(\omega^*(v) = 2k - 6 - 2\alpha - \beta \geq 2k - 6 - k = k - 6 \geq 0\).

Let \(f\) be a \(k\)-face.

**Case** \(k = 3\). Initially \(\omega(f) = -3\). By (C8), \(f\) is not a \((3, 3, 4^-)\)-face.

Let \(f = uvw\) be a \((3, 3, 5)\)-face so that \(d(u) = d(v) = 3\) and \(d(w) = 5\). By (C9) the outer neighbor of \(u\) (resp. \(v\)) has degree at least 5 and so gives at least 1 to \(f\) by R2. By R6, \(w\) gives 1 to \(f\). It follows that \(\omega^*(f) = -3 + 2 \cdot 1 + 1 = 0\).

Let \(f = uvw\) be a \((3, 3, 6^+)\)-face so that \(d(u) = d(v) = 3\) and \(d(w) = 6\). By (C6), the outer neighbor of \(u\) (resp. \(v\)) has degree at least 4 and so gives at least \(\frac{1}{2}\) to \(f\) by R1. By R7, \(w\) gives 2 to \(f\). It follows that \(\omega^*(f) = -3 + 2 \cdot \frac{1}{2} + 2 = 0\).

Let \(f = uvw\) be a \((3, 4, 4^-)\)-face so that \(d(u) = 3\) and \(d(v) = d(w) = 4\). By (C7) the outer neighbor of \(u\) has degree at least 5 and so gives 1 to \(f\) by R2. Vertices \(v\) (resp. \(w\)) give 1 to \(f\) by R3. Hence \(\omega^*(f) = -3 + 1 + 2 \cdot 1 = 0\).

Let \(f = uvw\) be a \((3, 4, 5^-)\)-face so that \(d(u) = 3, d(v) = 4\) and \(d(w) = 5\). By (C10), the outer neighbor of \(u\) has degree at least 4 and so gives at least \(\frac{1}{2}\) to \(f\) by R1. Vertices \(v\) and \(w\) give each 1 and \(\frac{1}{2}\) to \(f\) respectively by R3 and R5. Hence \(\omega^*(f) = -3 + \frac{1}{2} + 1 + \frac{1}{2} = 0\).
Let \( f = uvw \) be a \((3, 4, 6^+)\)-face so that \( d(u) = 3, d(v) = 4 \) and \( d(w) \geq 6 \). By R3 and R7, vertices \( v \) and \( w \) give each 1 and 2 to \( f \) respectively. Hence \( \omega^*(f) = -3 + 1 + 2 = 0 \).

Let \( f = uvw \) be a \((3, 5, 6^+)\)-face so that \( d(u) = 3, d(v) = 5, d(w) = 6 \). Assume \( f \) is poor and \( v \) is light. By (C5) \( w \) cannot be light. Hence \( \omega^*(f) = -3 + 1 + 2 = 0 \) by R4 and R6. Assume \( f \) is not poor. Then \( \omega^*(f) = -3 + 2 : \frac{1}{2} = 0 \) by R5.

Let \( f = uvw \) be a \((3, 5^+, 6^+)\)-face so that \( d(u) = 3, d(v) = 5, d(w) \geq 6 \). Vertices \( v \) and \( w \) give each at least 1 and 2 respectively by R6-7. Hence it appears less than 5 times, we can.

\[ k = 5 \times \left(3 + 1 + 2\right) = 0 \] by R5.

\[ k = 5 \times \left(3 + 1 + 2\right) = 0 \] by R5.

It follows that every vertex and face has a non-negative charge as required. This completes the proof.

4 (4, 0, 0)-colorability of \( \mathcal{F} \)

In this section we prove that every graph of \( \mathcal{F} \) is \((4, 0, 0)\)-colorable, namely we consider the case of \((s_1, s_2, s_3) = (4, 0, 0)\) for which \( s = s_1 + s_2 + s_3 = 4 \).

4.1 Reducible configurations for \((4, 0, 0)\)-coloring

In this section we study structural properties of \( G \) and establish a number of reducible configurations. See Figure 3.

A bad 8-vertex is a 8-vertex \( v \) incident to three \((3, 3, 8)\)-faces and to a \((3, 8, *\))-face \( f = uvw \) with \( d(u) = 3, d(v) = 8 \), where the vertex \( w \) is called the sponsor of \( f \) and \( f \) is a bad face of \( v \). See Figure 2.

![Figure 2: A bad 8-vertex \( v \) whose bad face is \( uvw \) with sponsor \( w \). (Drawing conventions are the same as in Figure 1.)](image)

\( (C1') \) \( G \) contains no 2\(^-\)-vertices.

\( (C2') \) For \( 8 \leq k \leq 10 \), a \( k \)-vertex cannot be incident to exactly \( k - 5 \) \((3, 3, k)\)-faces and adjacent to \( k \) 3-vertices.

\[ \text{Proof:} \] Suppose \( v \) is a \( k \)-vertex incident to exactly \( k - 5 \) \((3, 3, k)\)-faces and adjacent to \( 10 - k \) other 3-vertices not in these \((3, 3, k)\)-faces. By the minimality of \( G \), the graph \( G' \) obtained from \( G \) by deleting \( v \) and all its neighbors admits a \((4, 0, 0)\)-coloring. We color properly and sequentially all neighbors of \( v \). Since each \((3, 3, k)\)-face contains at most one vertex colored by 1, color 1 appears at most \( k - 5 + 10 - k = 5 \) times on the neighbors of \( v \). If it appears less than 5 times, we can
color \( v \) with 1, a contradiction. Hence color 1 appears exactly 5 times, once in each \((3,3,k)\)-face and on all the \(10 - k\) other 3-vertices. For each \((3,3,k)\)-face \( vxy \) with \( d(x) = d(y) = 3 \), where \( x \) is colored by 1, \( y \) is colored by 2 or 3. In the case of \( y \) is colored by 3, if the outer neighbor of \( y \) is colored by 1 (resp. 2), then we can recolor \( y \) by 2 (resp. 1). Then we can color \( v \) with 3 to obtain a \((4,0,0)\)-coloring of \( G \), a contradiction. □

(C3') Every 3-vertex of \( G \) is adjacent to at least one \( 7^+ \)-vertex.

Proof. This follows from the fact that the degree sequence for the three neighbors of a 3-vertex is lexicographically at least \((7,3,3)\) by Lemma 5. □

(C4') If \( uvw \) is a \((3,3,7)\)-face with \( d(u) = 3 \), then the outer neighbor of \( u \) has degree at least 4.

Proof. Suppose to the contrary that \( G \) has a \((3,3,7)\)-face \( uvw \) with \( d(u) = d(v) = 3 \) and \( d(w) = 7 \), but the outer vertex \( x \) of \( u \) has \( d(x) = 3 \). By Lemma 5, the degree sequence for the three neighbors of \( u \) is lexigraphically at least \((7,3,3)\). Hence \( w \) is colored by 1 and \( v \) is colored by 2 or 3. This contradicts the last sentence of Lemma 5 as \( w \) is adjacent to \( v \). □

(C5') The sponsor \( w \) of a bad 8-vertex \( v \) has degree at least 8 and is not a bad 8-vertex.

Proof. Suppose to the contrary that the bad 8-vertex \( v \) is incident to three \((3,3,8)\)-faces \( x_1x_2v, y_1y_2v \) and \( z_1z_2v \) and to a \((3,8,*)*\)-face \( uvw \) with \( d(u) = 3 \) and \( 3 \leq d(w) \leq 7 \) or \( w \) a bad 8-vertex. By the minimality of \( G \), the graph \( G' = G - \{v, x_1, x_2, y_1, y_2, z_1, z_2, u\} \) admits a \((4,0,0)\)-coloring. We can assume that \( w \) is nicely colored; otherwise, if \( d(w) \leq 7 \), then we can recolor it properly, and if \( w \) is a bad 8-vertex, then we can recolor properly all its colored neighborhood and then color \( w \) nicely. Now we color properly and sequentially \( x_1, x_2, y_1, y_2, z_1, z_2, u \), and we assign color 1 to \( v \) (color 1 appears at most 4 times on the neighbors of \( v \)). This extends the \((4,0,0)\)-coloring to \( G \), a contradiction. □

4.2 Discharging procedure for \((4,0,0)\)-coloring

We now apply a discharging procedure to reach a contradiction. The discharging rules are as follows:

R1'. For \( 4 \leq k \leq 6 \), every \( k \)-vertex gives \( \frac{1}{2} \) to each pendant 3-face.

R2'. Every \( 7^+ \)-vertex gives 1 to each pendant 3-face.

R3'. For \( 4 \leq k \leq 6 \), every \( k \)-vertex gives 1 to each incident 3-face.

R4'. Every \( 7^+ \)-vertex gives 1 to each incident \((4^+,4^+,4^+)\)-face.

R5'. Every non-bad \( 7^+ \)-vertex gives 2 to each incident \((4^+,4^+,4^+)\)-face; every bad 8-vertex gives 1 to its bad 3-face.

R6'. Every 7-vertex gives 2 to each incident \((3,3,7)\)-face.

R7'. For \( k \geq 8 \), every \( k \)-vertex gives 3 to each incident \((3,3,k)\)-face.

Let \( v \) be a \( k \)-vertex with \( k \geq 3 \) by (C1'). Initially \( \omega(v) = 2k - 6 \).

Case \( k = 3 \). The discharging procedure does not involves 3-vertices. Hence \( \omega^*(v) = \omega(v) = 0 \).

Case \( 4 \leq k \leq 6 \). Vertex \( v \) gives 1 to each of the \( \alpha \) incident 3-faces by R3' and \( \frac{1}{2} \) to each of the \( \beta \) pendant 3-faces by R1'. By Observation 2, \( \omega^*(v) \geq 2k - 6 - (\alpha + \frac{1}{2}\beta) \geq 2k - 6 - \frac{1}{2}k = \frac{3}{2}k - 6 \geq 0 \).

Case \( k = 7 \). Vertex \( v \) gives 2 to each of the \( \alpha' \) incident \((3,3,+)\)-faces by R5' and 1 to each of the \( \alpha'' \) incident \((4^+,4^+,4^+)\)-faces by R4', and 1 to each of the \( \beta \) pendant 3-faces by R2'. By Observation 2, \( \omega^*(v) \geq 2k - 6 - (2\alpha' + \alpha'' + \beta) \geq 2k - 6 - k = k - 6 > 0 \).

Case \( k \geq 8 \). For the case when \( v \) is a bad 8-vertex, \( v \) gives 3 to each incident \((3,3,8)\)-face by R7' and 1 to the bad 3-face by R5'. Hence \( \omega^*(v) = 2 \cdot 8 - 6 - 3 \cdot 3 - 1 = 0 \).
Figure 3: The reducible configurations (C2')-(C5'). (Drawing conventions are the same as in Figure 1.)
Now assume that $v$ is not a bad 8-vertex. By R7', R5', R4' and R2', $v$ gives 3 to each of the $\alpha'$ incident $(3,3,k)$-faces, 2 to each of the $\alpha''$ incident $(3,4^+,4^+)$-faces, 1 to each of the $\alpha'''$ incident $(4^+,4^+,4^+)$-faces, and 1 to each of the $\beta$ pendent 3-faces. By Observation 2, $\omega^*(v) = 2k - 6 - (3\alpha' + 2\alpha'' + \alpha''' + \beta) \geq 2k - 6 - \left\lfloor \frac{3k}{2} \right\rfloor - 6 \geq 0$ except for the cases (1) $k = 10$ with $\alpha' = 5$, (2) $k = 9$ with $\alpha' = 4$ and $\beta = 1$, (3) $k = 8$ with $\alpha' = 3$ and $\beta = 2$ (note that the bad 8-vertex case, i.e. $\alpha' = 4$ or $\alpha' = 3$ with $\alpha'' = 1$, is excluded). The exceptional cases give a $k$-vertex, $8 \leq k \leq 10$, with exactly $k - 5 (3,3,k)$-faces and adjacent only to 3-vertices, a contradiction to (C2').

Let $f$ be a $k$-face.

**Case** $k = 3$. Initially $\omega(f) = -3$.

Let $f =uvw$ be a $(a_1,a_2,a_3)$-face with $3 \leq a_1 \leq 6, 3 \leq a_2 \leq 6$ and $3 \leq a_3 \leq 6$. By (C3'), the outer neighbor of each 3-vertex incident to $f$ has degree at least 7 and gives each at least 1 to $f$ by R2'. By R3', each d-vertex with $4 \leq d \leq 6$ incident to $f$ gives 1 to $f$. It follows that $\omega^*(f) = -3 + 3 = 0$.

Let $f =uvw$ be a $(3,3,7)$-face so that $d(u) = d(v) = 3$ and $d(w) = 7$. By (C4') the outer neighbor of $u$ (resp. $v$) has degree at least 4 and so gives at least $\frac{1}{2}$ to $f$ by R1'. By R6', $w$ gives 2 to $f$. It follows that $\omega^*(f) = -3 + 2 \cdot \frac{1}{2} + 2 = 0$.

Let $f =uvw$ be a $(3,3,8^+)$-face so that $d(u) = d(v) = 3$ and $d(w) \geq 8$. By R7', $w$ gives 3 to $f$. It follows that $\omega^*(f) = -3 + 3 = 0$.

Let $f =uvw$ be a $(3,4^+,7^+)$-face so that $d(u) \geq 3, d(v) \geq 4$ and $d(w) \geq 7$. By R3'-$5^*$, vertices $u$ and $v$ gives at least 3 to $f$ and so $\omega^*(f) = -3 + 3 = 0$, except for the case when $f$ is a bad 3-face with the pair $v,w$ being either two bad 8-vertices or a bad 8-vertex and a 6$^-$-vertex. But these two exceptional cases are impossible by (C5').

Finally, let $f =uvw$ be a $(4^+,4^+,4^+)$-face. Every incident vertex gives at least 1 to $f$ by R3'-$4^*$. Hence $\omega^*(f) \geq 0$.

It follows that every vertex and face has a non-negative charge as required. This completes the proof.

**References**


