18.

Sol.

$$2xy' + y = 6x \Rightarrow y' + \frac{1}{2x}y = 3$$

The integrating factor:

$$\begin{aligned} I(x) &= e^{\int P(x)dx} \\ &= e^{\int \frac{1}{2x}dx} \\ &= e^{\frac{1}{2}\ln|x|} \\ &= e^{\frac{1}{2}\ln x} \\ &= \sqrt{x}. \end{aligned}$$

Multiply the integrating factor to the differential equation, we have:

$$\sqrt{x}y' + \frac{1}{2\sqrt{x}}y = 3\sqrt{x}$$

i.e.

$$(\sqrt{x}y)' = 3\sqrt{x}$$

$$\begin{aligned} \Rightarrow \sqrt{x}y &= \int 3\sqrt{x}dx \\ &= 3\cdot(\frac{2}{3}x^{\frac{3}{2}}) + C \\ &= 2x^{\frac{3}{2}} + C \end{aligned}$$

Therefore, we get:

$$\begin{aligned} y(x) &= 2x + \frac{C}{\sqrt{x}} \\ y(4) &= 8 + \frac{2}{C} \;\; \Rightarrow \;\; C = 24 \\ y(x) &= 2x + 24\frac{1}{\sqrt{x}} \end{aligned}$$

Q.E.D.

20.

Sol.

Let $u(x) = y(x) - 1$, we have

$$\frac{du}{dx} = \frac{dy}{dx}$$

$$y(x) = u(x) + 1$$

We have

$$(x^2+1)\frac{du}{dx} + 3x \cdot u(x) = 0,\ u(0) = 1$$

The integrating factor is

$$e^{\int \frac{3x}{x^2+1}dx} = e^{\ln(x^2+1)^{\frac{3}{2}}} = (x^2+1)^{\frac{3}{2}}$$

Therefore,

$$(x^2+1)^{\frac{3}{2}}\frac{du}{dx} + 3x(x^2+1)^{\frac{1}{2}}u(x) = 0$$

$$((x^2+1)^{\frac{3}{2}}u(x))' = 0$$

$$\Rightarrow (x^2+1)^{\frac{3}{2}}u(x) = C$$

$$u(x) = C \cdot (x^2+1)^{\frac{-3}{2}} = \frac{C}{(\sqrt{x^2+1})^3}$$

Since $u(0) = 1 \Rightarrow C = 1$

$$u(x) = \frac{1}{(\sqrt{x^2+1})^3}$$

$$\Rightarrow y(x) = u(x) + 1 = \frac{1}{(\sqrt{x^2+1})^3} + 1$$

Q.E.D.

22.

Sol.

The integrating factor

$$e^{\int \cos x dx} = e^{\sin x}$$

So we have

$$e^{\sin x} y' + \cos x e^{\sin x} y = \cos x e^{\sin x}$$

i.e.

$$(e^{\sin x} y)' = \cos x e^{\sin x}$$

$$\Rightarrow e^{\sin x} y = \int \cos x e^{\sin x} dx = e^{\sin x} + C$$

Therefore, $y = y(x) = 1 + C\dot{e}^{-\sin x}$, C is a constant. Q.E.D.

24.

Sol.

$$xy' + y = -xy^2 \Rightarrow y' + \frac{1}{x} y = -y^2$$

Let $u = u(x) = y^{1-2} = y^{-1}$, we have

$$\frac{du}{dx} + \frac{-1}{x} u = 1$$

The integrating factor is

$$e^{\int \frac{-1}{x} dx} = e^{-\ln|x|} = \frac{1}{x}$$

So we have the differential equation:
$\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = \frac{1}{x}$
$\Rightarrow (\frac{1}{x}u)' = \frac{1}{x}$
$\Rightarrow u(x) = x(\int \frac{1}{x}dx) = x(\ln|x| + C)$
$\Rightarrow y(x) = \frac{1}{u(x)} = \frac{1}{x(\ln|x| + C)}$

Q.E.D.

26.

Sol.

Let $u(x) = y'(x)$, then we have

$$x \cdot \frac{du}{dx} + 2u = 12x^2 \quad \Rightarrow \frac{du}{dx} + \frac{2}{x}u = 12x$$

The integrating factor is $e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = x^2$, so we have

$$x^2\frac{du}{dx} + 2xu = 12x^3$$

i.e. $(x^2u)' = 12x^3$
$\Rightarrow x^2u = \int 12x^3dx = 3x^4 + C_1$
$\Rightarrow u = u(x) = 3x^2 + C_1x^{-2}$
$\Rightarrow y(x) = \int u(x)dx = \int(3x^2 + C_1x^{-2})dx = x^3 - C_1 \cdot x^{-1} + C_2$.
where C_1 , C_2 are constants.

Q.E.D.

28.

Sol.

Use the equation 4 in textbook, we have a differential equation:

$$1 \cdot \frac{dI}{dt} + 20 \cdot I = 40\sin(60t)$$

where $I = I(t)$ is a function of t.
And the integrating factor is $e^{\int 2dt} = e^{20t}$, we have

$$e^{20t}\frac{dI}{dt} + 20e^{20t}I = 40e^{20t}\sin(60t)$$

i.e.
$$(e^{20t}I)' = 40e^{20t}\sin(60t)$$

By using the technique of integration by parts, we have

$$e^{20t}I = \frac{1}{5}e^{20t}\sin(60t) - \frac{3}{5}e^{20t}\cos(60t) + C$$

$$\Rightarrow I = I(t) = \frac{1}{5}\sin(60t) - \frac{3}{5}\cos(60t) + C \cdot e^{-20t}$$

Since $I(0) = \frac{-3}{5} + C = 1$, $C = \frac{8}{5}$
$\Rightarrow I(t) = \frac{1}{5}\sin(60t) - \frac{3}{5}\cos(60t) + \frac{8}{5} \cdot e^{-20t}$.

Q.E.D.

30.
Sol.
Use the result in Exercise 29, we have a differential equation:

$$R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t)$$

Since $R = 2$, $C = 0.01$, $E(t) = 10\sin(60t)$, the corresponding equation is

$$2 \cdot \frac{dQ(t)}{dt} + \frac{1}{0.01}Q(t) = 10\sin(60t)$$

$$\Rightarrow \frac{dQ(t)}{dt} + 50Q(t) = 5\sin(60t)$$

Solve it by using the integrating factor e$^{\int 50dt} = e^{50t}$,
$e^{50t}\frac{dQ}{dt} + 50e^{50t}Q(t) = 5e^{50t}\sin(60t)$

i.e.

$$(e^{50t}Q(t))' = 5e^{50t}\sin(60t)$$

$$\Rightarrow e^{50t}Q(t) = \int 5e^{50t}\sin(60t)dt$$
$$= \frac{1}{122}(5\sin(60t) - 6\cos(60t))e^{50t} + C$$

$$\Rightarrow Q(t) = \frac{1}{122}(5\sin(60t) - 6\cos(60t)) + Ce^{-50t}$$

Since $Q(0) = \frac{-6}{122} + C = 0 \Rightarrow C = \frac{6}{122}$, we solve the charge function:

$$Q(t) = \frac{1}{122}(5\sin(60t) - 6\cos(60t)) + \frac{6}{122}e^{-50t}$$

And the current $I(t) = \dfrac{dQ(t)}{dt} = \dfrac{150\cos(60t) + 180\sin(60t) - 150e^{-50t}}{61}$.

Q.E.D.

32.
Sol.
Solve the equation in Exercise 31 to get the solution:

$$P(t) = M + Ce^{-kt}, k > 0$$

For $P(0) = 0$, $P(t) = M[1 - e^{-kt}]$.

For the first worker, $P_1(1) = 25$, $P_1(2) = 45$, i.e.

$$M_1[1 - e^{-k_1}] = 25$$

$$M_1[1 - e^{-2k_1}] = 45$$

$\Rightarrow \dfrac{1-e^{-k_1}}{1-e^{-2k_1}} = \dfrac{25}{45} = \dfrac{5}{9}$
$\Rightarrow e^{-k_1} = \dfrac{4}{5}$
Then we can evaluate $M_1 = 25 \cdot \dfrac{1}{1-\frac{4}{5}} = 125$, which is the maximum number of units per hour the first man can achieve.

Similarly, we can get $M_2 = 61.25$.

Q.E.D.

34.
Sol.
Let $y(t)$ be the amount of chlorine in the tank at time t , and then $y(0) = 20$ (g). The amount of liquid in the tank at time t is $(400-6t)$ (L), thus the concentration of chlorine at time t is $\dfrac{y(t)}{400-6t}$ $(\frac{g}{L})$. The rate of chlorine leaving the tank at time t is $\dfrac{y(t)}{400-6t} \cdot 10$ since there's 10 (L) liquid leaves the tank per second.
Therefore, we have a differential equation:

$$\frac{dy}{dt} = \frac{-10y(t)}{400-6t} \qquad = \frac{-5y(t)}{200-3t}$$

Rearrange the differential equation, we have

$$\frac{dy}{y} = \frac{-5dt}{200-3t}$$

Integrate both side of the equation separately, we have

$$\ln y = \frac{5}{3}\ln(200-3t) + C$$

$$\Rightarrow y(t) = e^{\frac{5}{3}\ln(200-3t)+C}$$
$$= e^{C} \cdot (200-3t)^{\frac{5}{3}}$$

Since $y(0) = e^C \cdot 200^{\frac{5}{3}} = 20$, $e^C = \dfrac{20}{200^{\frac{5}{3}}}$

We have the solution:

$$y(t) = \frac{20}{200^{\frac{5}{3}}} \cdot (200 - 3t)^{\frac{5}{3}}$$

.

Q.E.D.

36.

Sol.

Use the equation in Exercise 35(a) :

$$v = \frac{mg}{c}(1 - e^{\frac{-ct}{m}})$$

Differentiate it with respect to m ,we have:

$$\begin{aligned}
\frac{dv}{dm} &= \frac{g}{c} - (\frac{g}{c}e^{\frac{-ct}{m}} + \frac{gt}{m}e^{\frac{-ct}{m}}) \\
&= \frac{g}{c}(1 - e^{\frac{-ct}{m}} - \frac{ct}{m}e^{\frac{-ct}{m}}) \\
&= \frac{g}{c}(1 - e^{-u} - ue^{-u}) \\
&= \frac{g}{c}(1 - \frac{1+u}{e^u})
\end{aligned}$$

where $u = \dfrac{ct}{m} > 0, \forall m > 0, t > 0$

Since $e^u > 1 + u, \forall u > 0$, $\dfrac{1+u}{e^u} < 1, \forall u > 0$

$$\Rightarrow (1 - \frac{1+u}{e^u}) > 0$$

$$\Rightarrow \frac{dv}{dm} = \frac{g}{c}(1 - \frac{1+u}{e^u}) > 0, \forall u > 0, t > 0$$

That is, v increases as m increases.

Q.E.D.