[Section 8.2] Area of a surface of revolution

1. $y = x^4$, $0 \leq x \leq 1 \Rightarrow \frac{dy}{dx} = 4x^3$

   (a) about x-axis:
$$S = \int 2\pi y \ ds = \int_0^1 2\pi x^4 \sqrt{1+16x^6} \ dx$$
   (b) about y-axis:
$$S = \int 2\pi x \ ds = \int_0^1 2\pi x \sqrt{1+16x^6} \ dx \qquad \blacksquare$$

8. $y = \cos(2x)$, $0 \leq x \leq \frac{\pi}{6} \Rightarrow \frac{dy}{dx} = -2\sin(2x)$

$$\begin{aligned} S &= \int_0^{\pi/6} 2\pi \cos(2x)\sqrt{1+4\sin^2(2x)}\, dx \\ &= \int_0^{\sqrt{3}/2} \pi\sqrt{1+4u^2}\, du && by \quad u = \sin(2x) \\ &= \int_0^{\pi/3} \pi\sqrt{1+\tan^2\theta}\, \tfrac{\sec^2\theta}{2}\, d\theta && by \quad u = \tfrac{\tan\theta}{2} \\ &= \tfrac{\pi}{2}\int_0^{\pi/3} \sec^3(\theta)\, d\theta \\ &= \tfrac{\pi}{2}\left[\tfrac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|)\right]_0^{\pi/3} \\ &= \tfrac{\pi}{4}(2\sqrt{3} + \ln(2+\sqrt{3})) \qquad \blacksquare \end{aligned}$$

15. $x = \sqrt{a^2-y^2}$, $0 \leq y \leq \frac{a}{2} \Rightarrow \frac{dx}{dy} = \frac{-y}{\sqrt{a^2-y^2}}$

$$\begin{aligned} S &= \int_0^{a/2} 2\pi\sqrt{a^2-y^2}\, \sqrt{\left(\tfrac{-y}{\sqrt{a^2-y^2}}\right)^2+1}\, dy \\ &= \int_0^{a/2} 2\pi\sqrt{a^2-y^2}\, \sqrt{\tfrac{y^2}{a^2-y^2}+1}\, dy \\ &= \int_0^{a/2} 2\pi\sqrt{a^2-y^2}\, \tfrac{a}{\sqrt{a^2-y^2}}\, dy \\ &= \int_0^{a/2} 2\pi a\, dy \\ &= \pi a^2 \qquad \blacksquare \end{aligned}$$

25. $S = \int_1^\infty 2\pi y\sqrt{1+(y')^2}\, dx$, $y = \frac{1}{x}$

$$\begin{aligned} S &= 2\pi \int_1^\infty \tfrac{1}{x}\sqrt{1+\tfrac{1}{x^4}}\, dx \\ &= 2\pi \int_1^\infty \tfrac{\sqrt{x^4+1}}{x^3}\, dx \\ &> 2\pi \int_1^\infty \tfrac{\sqrt{x^4}}{x^3}\, dx && \forall x > 0 \\ &= 2\pi \int_1^\infty \tfrac{1}{x}\, dx && div. \end{aligned}$$

Hence, S diverge. $\blacksquare$

29. (a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ,\ a>b \quad\Rightarrow\quad \frac{dy}{dx}=-\frac{2x}{a^2}/\frac{2y}{b^2}=-\frac{b^2x}{a^2y}$

$$1+(\frac{dy}{dx})^2=1+\frac{b^4x^2}{a^4y^2}=1+\frac{b^2x^2}{a^2(a^2-x^2)}=\frac{a^4-(a^2-b^2)x^2}{a^2(a^2-x^2)}$$

The ellipsoid's surface area is twice the area generated by rotating the first-quadrant portion of the ellipse about the x-axis. Thus,

$$\begin{aligned}
S &= 2\int_0^a 2\pi y\sqrt{1+[dy/dx]^2}\,dx \\
&= 4\pi\int_0^a \tfrac{b}{a}\sqrt{a^2-x^2}\,\frac{\sqrt{a^4-(a^2-b^2)x^2}}{a\sqrt{a^2-x^2}}\,dx \\
&= \tfrac{4\pi b}{a^2}\int_0^a \sqrt{a^4-(a^2-b^2)x^2}\,dx \\
&= \tfrac{4\pi b}{a^2}\int_0^{\sqrt{a^2-b^2}/a} \sqrt{a^4-a^4u^2}\,\frac{a^2}{\sqrt{a^2-b^2}}\,du && let \quad a^2u=\sqrt{a^2-b^2}\,x \\
&= \frac{4\pi a^2b}{\sqrt{a^2-b^2}}\int_0^{\sqrt{a^2-b^2}/a} \sqrt{1-u^2}\,du \\
&= \frac{4\pi a^2b}{\sqrt{a^2-b^2}}\int_0^{\sin^{-1}(\sqrt{a^2-b^2}/a)} \cos^2\theta\,d\theta && let \quad u=\sin\theta \\
&= \frac{4\pi a^2b}{\sqrt{a^2-b^2}}\int_0^{\sin^{-1}(\sqrt{a^2-b^2}/a)} \frac{1+\cos(2\theta)}{2}\,d\theta \\
&= \frac{4\pi a^2b}{\sqrt{a^2-b^2}}[\frac{\theta}{2}+\frac{\sin(2\theta)}{4}]\Big|_0^{\sin^{-1}(\sqrt{a^2-b^2}/a)} && (remind \quad \sin(2\theta)=2\sin\theta\cos\theta) \\
&= \frac{2\pi a^2b}{\sqrt{a^2-b^2}}[\sin^{-1}u+u\cos(\sin^{-1}u)]\Big|_0^{\sqrt{a^2-b^2}/a} && by \quad \theta=\sin^{-1}u \\
&= \frac{2\pi a^2b}{\sqrt{a^2-b^2}}[\sin^{-1}u+u\sqrt{1-u^2}]\Big|_0^{\sqrt{a^2-b^2}/a} \\
&= \frac{2\pi a^2b}{\sqrt{a^2-b^2}}[\sin^{-1}(\frac{\sqrt{a^2-b^2}}{a})+\frac{\sqrt{a^2-b^2}}{a}\sqrt{1-\frac{a^2-b^2}{a^2}}] \\
&= 2\pi[\frac{a^2b}{\sqrt{a^2-b^2}}\sin^{-1}(\frac{\sqrt{a^2-b^2}}{a})+b^2] && \blacksquare
\end{aligned}$$

(b) Similarly, change a and b, we get

$$S=2\pi[\frac{ab^2}{\sqrt{b^2-a^2}}\sin^{-1}(\frac{\sqrt{b^2-a^2}}{b})+a^2] \quad \blacksquare$$

31. $S=\int_a^b 2\pi[c-f(x)]\sqrt{1+[f'(x)]^2}\ dx \quad \blacksquare$

36. $g(x)=f(x)+c \quad\Rightarrow\quad g'(x)=f'(x)$

$$\begin{aligned}
S_g &= \int_a^b 2\pi g(x)\sqrt{1+[g'(x)]^2}\,dx \\
&= \int_a^b 2\pi[f(x)+c]\sqrt{1+[f'(x)]^2}\,dx \\
&= \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx+2\pi c\int_a^b\sqrt{1+[f'(x)]^2}\,dx \\
&= S_f+2\pi cL && \blacksquare
\end{aligned}$$