SECTION 8.1

2. Use the arc length formula with $y = \sqrt{2 - x^2}$,

$$\frac{dy}{dx} = \frac{-x}{\sqrt{2-x^2}}.$$

Hence

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + (\frac{dy}{dx})^2} dx \\ &= \int_0^1 \sqrt{1 + \frac{x^2}{2-x^2}} dx \\ &= \int_0^1 \frac{\sqrt{2}}{\sqrt{2-x^2}} dx \\ &= \sqrt{2}[\sin^{-1}(\frac{x}{\sqrt{2}})]\Big|_0^1 \\ &= \frac{\sqrt{2}}{4}\pi. \end{aligned}$$

The curve is one-eight of the circle with radius $\sqrt{2}$, and the length is

$$\frac{1}{8}.2.\sqrt{2}\pi = \frac{\sqrt{2}}{4}\pi.$$

10. Differentiating the equation implicitly with respect to y,

$$\frac{dx}{dy} = \frac{1}{2}(y^3 - y^{-3}).$$

Hence

$$\begin{aligned} L &= \int_1^2 \sqrt{1 + (\frac{dx}{dy})^2} dy \\ &= \int_1^2 \sqrt{1 + \frac{1}{4}(y^6 - 2 + y^{-6})} dy \\ &= \int_1^2 \sqrt{\frac{1}{4}(y^6 + 2 + y^{-6})} dy \\ &= \frac{1}{2}\int_1^2 (y^3 + y^{-3}) dy \\ &= \frac{1}{2}[\frac{y^4}{4} + \frac{y^{-2}}{-2}]\Big|_1^2 \\ &= \frac{33}{16}. \end{aligned}$$

The length of the curve is $\frac{33}{16}$.

13. Differentiating the equation implicitly with respect to x,

$$\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x.$$

Hence

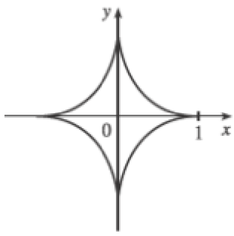
$$\begin{aligned} L &= \int_0^{\frac{\pi}{4}} \sqrt{1 + (\frac{dx}{dy})^2} dy \\ &= \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x} dy \\ &= \int_0^{\frac{\pi}{4}} |\sec x| dy \\ &= \ln(\sec x + \tan x)\Big|_0^{\frac{\pi}{4}} \\ &= \ln(\sqrt{2} + 1). \end{aligned}$$

31. $y^{2/3} = 1 - x^{2/3} \Rightarrow y = (1 - x^{2/3})^{3/2} \Rightarrow$

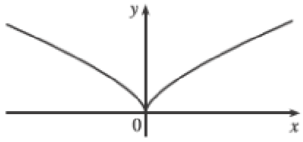
$\frac{dy}{dx} = \frac{3}{2}(1 - x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -x^{-1/3}(1 - x^{2/3})^{1/2} \Rightarrow$

$\left(\frac{dy}{dx}\right)^2 = x^{-2/3}(1 - x^{2/3}) = x^{-2/3} - 1$. Thus

$L = 4\int_0^1 \sqrt{1 + (x^{-2/3} - 1)}\, dx = 4\int_0^1 x^{-1/3}\, dx = 4 \lim_{t \to 0^+} \left[\frac{3}{2}x^{2/3}\right]_t^1 = 6.$



32. (a)



(b) $y = x^{2/3} \;\Rightarrow\; 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{2}{3}x^{-1/3}\right)^2 = 1 + \frac{4}{9}x^{-2/3}$. So $L = \int_0^1 \sqrt{1 + \frac{4}{9}x^{-2/3}}\,dx$ [an improper integral].

$x = y^{3/2} \;\Rightarrow\; 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{3}{2}y^{1/2}\right)^2 = 1 + \frac{9}{4}y$. So $L = \int_0^1 \sqrt{1 + \frac{9}{4}y}\,dy$.

The second integral equals $\frac{4}{9} \cdot \frac{2}{3}\left[\left(1 + \frac{9}{4}y\right)^{3/2}\right]_0^1 = \frac{8}{27}\left(\frac{13\sqrt{13}}{8} - 1\right) = \frac{13\sqrt{13} - 8}{27}$.

The first integral can be evaluated as follows:

$$\int_0^1 \sqrt{1 + \tfrac{4}{9}x^{-2/3}}\,dx = \lim_{t \to 0^+} \int_t^1 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}}\,dx = \lim_{t \to 0^+} \int_{9t^{2/3}}^9 \frac{\sqrt{u + 4}}{18}\,du \qquad \begin{bmatrix} u = 9x^{2/3}, \\ du = 6x^{-1/3}\,dx \end{bmatrix}$$

$$= \int_0^9 \frac{\sqrt{u + 4}}{18}\,du = \frac{1}{18} \cdot \left[\frac{2}{3}(u + 4)^{3/2}\right]_0^9 = \frac{1}{27}(13^{3/2} - 4^{3/2}) = \frac{13\sqrt{13} - 8}{27}$$

(c) $L =$ length of the arc of this curve from $(-1, 1)$ to $(8, 4)$

$$= \int_0^1 \sqrt{1 + \frac{9}{4}y}\,dy + \int_0^4 \sqrt{1 + \frac{9}{4}y}\,dy = \frac{13\sqrt{13} - 8}{27} + \frac{8}{27}\left[\left(1 + \frac{9}{4}y\right)^{3/2}\right]_0^4 \qquad \text{[from part (b)]}$$

$$= \frac{13\sqrt{13} - 8}{27} + \frac{8}{27}(10\sqrt{10} - 1) = \frac{13\sqrt{13} + 80\sqrt{10} - 16}{27}$$

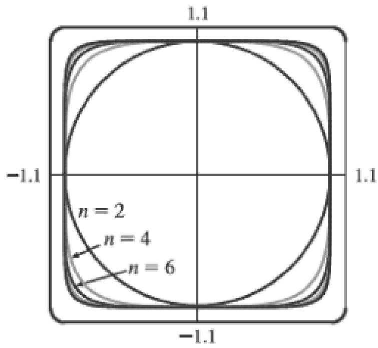
42. By symmetry, the length of the curve in each quadrant is the same, so we'll find the length in the first quadrant and multiply by 4.

$x^{2k} + y^{2k} = 1 \;\Rightarrow\; y^{2k} = 1 - x^{2k} \;\Rightarrow\; y = (1 - x^{2k})^{1/(2k)}$

(in the first quadrant), so we use the arc length formula with

$\frac{dy}{dx} = \frac{1}{2k}(1 - x^{2k})^{1/(2k)-1}(-2kx^{2k-1}) = -x^{2k-1}(1 - x^{2k})^{1/(2k)-1}$

The total length is therefore



$$L_{2k} = 4\int_0^1 \sqrt{1 + [-x^{2k-1}(1 - x^{2k})^{1/(2k)-1}]^2}\,dx = 4\int_0^1 \sqrt{1 + x^{2(2k-1)}(1 - x^{2k})^{1/k-2}}\,dx$$

Now from the graph, we see that as k increases, the "corners" of these fat circles get closer to the points $(\pm 1, \pm 1)$ and $(\pm 1, \mp 1)$, and the "edges" of the fat circles approach the lines joining these four points. It seems plausible that as $k \to \infty$, the total length of the fat circle with $n = 2k$ will approach the length of the perimeter of the square with sides of length 2. This is supported by taking the limit as $k \to \infty$ of the equation of the fat circle in the first quadrant: $\lim_{k \to \infty}(1 - x^{2k})^{1/(2k)} = 1$ for $0 \le x < 1$. So we guess that $\lim_{k \to \infty} L_{2k} = 4 \cdot 2 = 8$.