Solutions to 7.8

13. The integral exists. By definition, we pick $a = 0$. Then

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx$$

Note that

$$\lim_{c\to\infty}\int_0^c xe^{-x^2}dx = \lim_{c\to\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Big|_0^c = \lim_{c\to\infty}\left(\frac{1}{2}-\frac{1}{2}e^{-c^2}\right) = \frac{1}{2}$$

The opposite part is similar. We conclude the integral exists and is equal to 0.

16.

$$\int_a^{\pi/2} \sin 2\theta d\theta = -\frac{1}{2}\cos 2\theta\Big|_a^{\pi/2} = \frac{1}{2}(1+\cos 2a)$$

But the limit of the last term does not exist as $a \to -\infty$. So $\int_{-\infty}^{\pi/2} \sin 2\theta d\theta$ does not exist.

21. By changing variables,

$$\int_1^c \frac{\ln x}{x}dx = \int_1^c \ln x d(\ln x) = \frac{1}{2}(\ln x)^2\Big|_1^c = \frac{1}{2}(\ln c)^2 \to \infty \quad \text{as } c \to \infty$$

So the integral is not convergent.

28. Note that the integrand is not defined at $x = 3$. It is an improper integral.

$$\int_2^t \frac{dx}{\sqrt{3-x}} = -2\sqrt{3-x}\,\Big|_2^t = 2 - 2\sqrt{3-t} \to 2 \quad \text{as } t \to 3$$

Therefore, $\int_2^3 \frac{dx}{\sqrt{3-x}} = 2.$

37. Note that the integrand is not defined at $x = 0$. So consider

$$\int_{-1}^t \frac{1}{x^3}e^{1/x}dx = -\int_{-1}^t \frac{1}{x}e^{1/x}d\left(\frac{1}{x}\right) \tag{1}$$

Now let $y = 1/x$ make a substitution. Equation (1) becomes

$$-\int_{-1}^{-1/t} ye^y dy = (e^y - ye^y)\Big|_{-1}^{-1/t} = \frac{1}{t}e^{-1/t} + e^{-1/t} - \frac{2}{e} \to -\frac{2}{e} \quad \text{as } t \to 0$$

43. The graph of S is already given in the book. Here I just give the solution of area of S. Denote the area by $S(A)$. Then

$$S(A) = \int_{-\infty}^{\infty} \frac{2}{x^2+9}dx = 2\int_0^{\infty} \frac{2}{x^2+9}dx$$

Note the last equality holds if the last integral exists. Now

$$\int_0^a \frac{2}{x^2+9}dx = \frac{2}{3}\tan^{-1}\frac{x}{3}\Big|_0^a = \frac{2}{3}\tan^{-1}\frac{a}{3} \to \frac{\pi}{3} \quad \text{as } a \to \infty$$

The integral actually exists. So we conclude $A(S) = \frac{2}{3}\pi$

49. Note the obvious inequality $0 \leq \dfrac{x}{x^3+1} \leq \dfrac{1}{x^2}$ holds for all $x \geq 1$. Hence

$$\int_0^\infty \frac{x}{x^3+1}dx = \int_0^1 \frac{x}{x^3+1}dx + \int_1^\infty \frac{x}{x^3+1}dx \leq \int_0^1 \frac{x}{x^3+1}dx + \int_1^\infty \frac{1}{x^2}dx$$

Since $\dfrac{x}{x^3+1}$ is a positive continuous function on $[0,1]$. Hence integrable. We conclude the integral converges. That is,

$$\int_0^\infty \frac{x}{x^3+1}dx \leq \int_0^1 \frac{x}{x^3+1}dx + \int_1^\infty \frac{1}{x^2}dx < \infty$$

54. Clearly $|\sin^2 x| \leq 1$ for all x. The integral exists since

$$\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}dx \leq \int_0^\pi \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\,\Big|_0^\pi = 2\sqrt{\pi} < \infty$$

57. If $p < 1$, integrate the function directly. This leads to

$$\int_0^1 \frac{1}{x^p}dx = \frac{x^{1-p}}{1-p}\Big|_0^1 = \frac{1}{1-p}$$

If $p \geq 1$, consider the inequality $\dfrac{1}{x^p} \geq \dfrac{1}{x}$. This inequality holds since $x \in (0,1]$. Now by comparison of the integrands, we have

$$\int_t^1 \frac{1}{x^p}dx \geq \int_t^1 \frac{1}{x}dx \to \infty \quad \text{as } t \to 0$$

Therefore, by comparison, the integral is divergent.

58. Again by changing variables, let $y = \ln x$. We have

$$\int_e^\infty \frac{1}{x(\ln x)^p}dx = \int_1^\infty \frac{1}{y^p}dy$$

This is the reverse problem of 57. If $p > 1$, then the integral exists since

$$\int_1^\infty \frac{1}{y^p}dy = \lim_{t\to\infty}\int_1^t \frac{1}{y^p}dy = \lim_{t\to\infty}\frac{y^{1-p}}{1-p}\Big|_1^t = \frac{1}{p-1}$$

On the other hand, if $p \leq 1$, then $\dfrac{1}{y} \leq \dfrac{1}{y^p}$ for all $y \geq 1$. Hence

$$\int_1^t \frac{1}{y^p}dy \geq \int_1^t \frac{1}{y}dy = \ln y\Big|_1^t = \ln t$$

letting $t \to \infty$, then $\ln t \to \infty$. Thus the integral diverges for all $p \geq 1$.

61. (a) For any fixed a, compute

$$\int_a^\infty xdx = \lim_{c\to\infty}\int_a^c xdx = \frac{1}{2}x^2\Big|_a^c = \frac{1}{2}(c^2 - a^2) \to \infty \quad \text{as } c \to \infty$$

Hence the improper integral $\displaystyle\int_a^\infty xdx$ does not exist. So does $\displaystyle\int_{-\infty}^a xdx$. This implies $\displaystyle\int_{-\infty}^\infty xdx$ does not exist since it can't be defined.

(b) Clearly, $\int_{-t}^{t} xdx = 0$ for all t since the integrand is an odd function. So

$$\lim_{t\to\infty} \int_{-t}^{t} xdx = 0$$

Therefore, $\int_{-\infty}^{\infty} xdx \neq \lim_{t\to\infty} \int_{-t}^{t} xdx.$

76. For the left hand side, it's sum of all vertical lines. And the right hand side is sum of the horizontal lines on the same region. Hence they represent the same region. More precisely, the left hand side is the limit of the Riemann sum on x-axis, while the right hand side is the limit of the Riemann sum on y-axis.

$$\int_0^{\infty} e^{-x^2} dx = \int_0^1 \sqrt{-\ln y}\; dy$$

78. For any fixed $a > 0$,

$$\int_0^a \frac{x}{x^2+1} - \frac{C}{3x+1} dx = \frac{1}{2}\ln(x^2+1)\Big|_0^a - \frac{C}{3}\ln(3x+1)\Big|_0^a = \frac{1}{2}\ln(a^2+1) - \frac{C}{3}\ln(3a+1)$$

By the property of logarithm, the last term is equal to $\ln\left(\frac{\sqrt{a^2+1}}{\sqrt[3]{(3a+1)^C}}\right)$. Since the integral converges, this forces the limit

$$\lim_{a\to\infty} \ln\left(\frac{\sqrt{a^2+1}}{\sqrt[3]{(3a+1)^C}}\right)$$

exists. Hence we conclude from here that $C = 3$ and the limit is $\frac{1}{3}$.