Chapter 7.7 (Solution)

5. *sol.*

$f(x) = x^2 \sin x$, $\Delta x = \dfrac{\pi - 0}{8} = \dfrac{\pi}{8}$,

(a) Use the Midpoint Rule,

$$M_8 = \int_0^{\pi} x^2 \sin x \, dx \approx \frac{\pi}{8}\left[f(\frac{\pi}{16}) + f(\frac{3\pi}{16}) + \cdots f(\frac{15\pi}{16}) \right] \approx 5.9329566$$

(b) Use the Simpson's Rule, $S_8 = \displaystyle\int_0^{\pi} x^2 \sin x \, dx$

$\approx \dfrac{1}{3} \cdot \dfrac{\pi}{8} \left[f(0) + 4f(\frac{\pi}{8}) + 2f(\frac{2\pi}{8}) + 4f(\frac{3\pi}{8}) + 2f(\frac{4\pi}{8}) + 4f(\frac{5\pi}{8}) + 2f(\frac{6\pi}{8}) + 4f(\frac{7\pi}{8}) + f(\pi) \right]$

≈ 5.8692468

(c) Actual value: $\displaystyle\int_0^{\pi} x^2 \sin x \, dx = -x^2 \cos x \Big|_0^{\pi} + 2 \int_0^{\pi} x \cos x \, dx$

$= \pi^2 + 2 \left[x \sin x \Big|_0^{\pi} - \displaystyle\int_0^{\pi} \sin x \, dx \right] = \pi^2 + 2 \left[0 - \left(- \cos x \Big|_0^{\pi} \right) \right]$

$= \pi^2 - 4 \approx 5.8696044$

Error: $E_M = (\pi^2 - 4) - M_8 \approx -0.063352199$; $E_S = (\pi^2 - 4) - S_8 \approx 0.000357601$.

23. *sol.*

(a) $f(x) = e^{\cos x}$, $f'(x) = e^{\cos x}(-\sin x)$, $f''(x) = e^{\cos x}(\sin^2 x - \cos x)$. From the graph of $f''(x)$ (See Figure 1), we find that the maximum value of $|f''(x)|$ occurs at the endpoint of the interval $[0, 2\pi]$. Because of $f''(0) = -e$, we choose $K = e$ and $|f''(x)| \le K$.

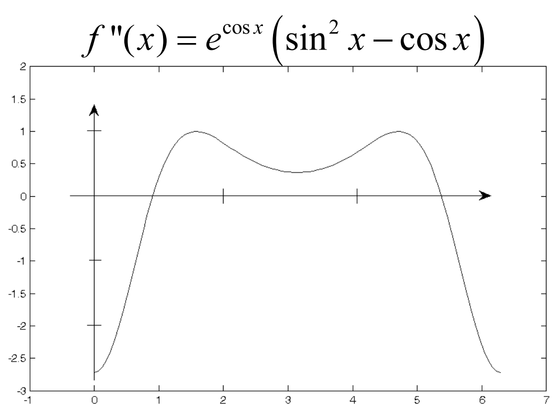

Figure 1: 7.7 Ex.23(a)

(b) $\Delta x = \dfrac{2\pi - 0}{10} = \dfrac{\pi}{5}$, $I = \displaystyle\int_0^{2\pi} f(x) \, dx$,

$$M_{10} = \Delta x \left[f(\frac{\pi}{10}) + f(\frac{3\pi}{10}) + \cdots + f(\frac{19\pi}{10}) \right] \approx 7.954926518$$

(c) $|E_M| \leq \dfrac{K(b-a)^3}{24n^2} = \dfrac{e \cdot (2\pi - 0)^3}{24 \cdot 10^2} \approx 0.280945995$

(d) $I = \displaystyle\int_0^{\pi} x^2 \sin x \, dx \approx 7.954926521$. (using software like CAS or Maple or ...)

(e) The actual error is about 3×10^{-9}; the estimate error in (c) is about 0.28. So the actual error is much smaller.

(f) $f'''(x) = e^{\cos x}(-\sin^3 x + 3\sin x \cos x + \sin x)$, and $f^{(4)}(x) = e^{\cos x}(\sin^4 x - 6\sin^2 x \cos x - 7\sin^2 x + \cos x + 3)$. From the graph of $f^{(4)}(x)$ (See Figure 2), we find that the maximum value of $|f^{(4)}(x)|$ occurs at the endpoint of the interval $[0, 2\pi]$. Because of $f^{(4)}(0) = 4e$, we choose $K = 4e$ and $|f^{(4)}(x)| \leq K$.

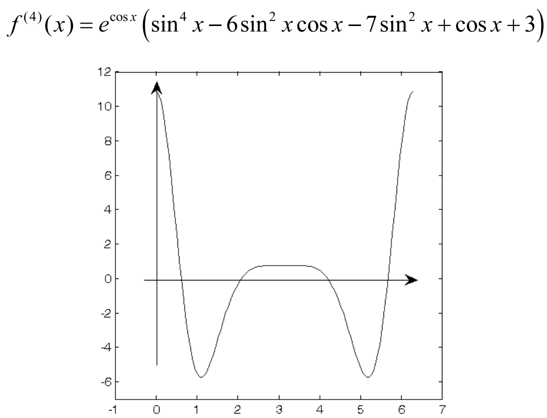

Figure 2: 7.7 Ex.23(f)

(g) $\Delta x = \dfrac{2\pi - 0}{10} = \dfrac{\pi}{5}$, $I = \displaystyle\int_0^{2\pi} f(x)\, dx$,

$$S_{10} = \frac{\Delta x}{3}\left[f(0) + 4f(\frac{\pi}{5}) + 2f(\frac{2\pi}{5}) + \cdots + 4f(\frac{9\pi}{5}) + f(2\pi)\right] \approx 7.953789422$$

(h) $|E_S| \leq \dfrac{K(b-a)^5}{180n^4} = \dfrac{4e \cdot (2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059153618$

(i) The actual error is about $7.954926521 - 7.953789422 \approx 0.001137099$; the estimate error in (h) is about 0.059153618. So the actual error is smaller.

(j) We require $|E_S| = \dfrac{K(b-a)^5}{180n^4} \leq 0.0001 \Rightarrow n^4 \geq \dfrac{K(b-a)^5}{180 \cdot 0.0001} = \dfrac{4e(2\pi)^5}{180 \cdot 0.0001} \approx 5915361.766 \Rightarrow n^4 \geq 5915362 \Rightarrow n \geq 49.3$. So we require $n \geq 50$ to guarantee that $|E_S| \leq 0.0001$.

45. *sol.*

We divide the interval $[a, b]$ into n intervals: $[x_{i-1}, x_i]$, for $i = 1, 2, \cdots, n$ when we use trapezoidal rule and midpoint rule, so we only consider one interval as shown in Figure 3. Let A, B, and E be x_{i-1}, x_i, and $\overline{x}_i$; the red curve represents $f(x)$ ($f''(x) <$

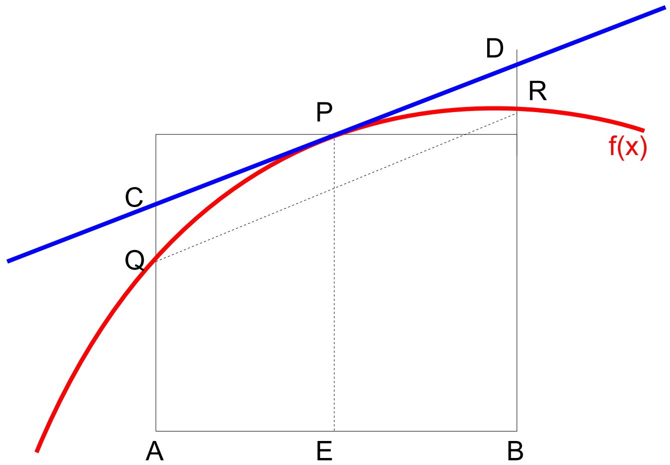

Figure 3: 7.7 Ex.45

0). The actual value of $\int_a^b f(x)\,dx$ is the area between the red curve and $\overline{AB}$. T_n is the area of trapezoid $AQRB$, so $T_n < \int_a^b f(x)\,dx$. M_n is the area of the rectangle with length $\overline{AB}$ and width $\overline{EP}$, and it equals the area of the trapezoid $ACDB$, where $\overline{CD}$ is the tangent line of $f(x)$ at point P. Therefore, $M_n > \int_a^b f(x)\,dx$.

46. *sol.*

Let $f(x)$ be a polynomial of degree < 3, so assume $f(x) = Ax^3 + Bx^2 + Cx + D$. Consider that we divide $[a, b]$ into two subintervals ($n = 2$), if we prove that the estimate is exact for $n = 2$, then for the general case (n is a large even number), the estimate is the sum of $n/2$ exact values, so it is a exact value.

Assume that $x_0 = -h$, $x_1 = 0$, and $x_2 = h$, the Simpson's rule gives the approximation value: $\displaystyle\int_{-h}^{h} f(x)\,dx \approx \frac{h}{3}\left[f(-h) + 4f(0) + f(h)\right]$

$= \dfrac{h}{3}\left[(-Ah^3 + Bh^2 - Ch + D) + 4D + (Ah^3 + Bh^2 + Ch + D)\right]$

$= \dfrac{2}{3}Bh^3 + 2Dh.$

The actual value of the integral is: $\displaystyle\int_{-h}^{h} f(x)\,dx = \int_{-h}^{h} (Ax^3 + Bx^2 + Cx + D)\,dx$

$= \dfrac{1}{4}Ax^4 + \dfrac{1}{3}Bx^3 + \dfrac{1}{2}x^2 + Dx\Big|_{-h}^{h}$

$= \dfrac{2}{3}Bh^3 + 2Dh.$

47. *sol.*

Suppose we divide $[a, b]$ into n subintervals with endpoints x_0, x_1, $\cdots$ x_n ($\Delta x = \dfrac{b-a}{n}$), and $\overline{x}_i = \dfrac{1}{2}(x_{i-1} + x_i)$ is the midpoint of $[x_{i-1}, x_i]$ for $i = 1, \cdots, n$. Therefore,

$$M_n = \Delta x \cdot \sum_{i=1}^{n} f(\overline{x}_i),$$
$$T_n = \frac{1}{2} \cdot \Delta x \cdot \left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right],$$
$$T_{2n} = \frac{1}{2} \cdot \left(\frac{\Delta x}{2}\right) \left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + 2\sum_{i=1}^{n} f(\overline{x}_i) + f(x_n) \right]$$
Then, $\frac{1}{2}[T_n + M_n] = T_{2n}$.

48. *sol.*

Suppose we divide $[a, b]$ into n subintervals with endpoints x_0, x_1, $\cdots$ x_n ($\Delta x = \frac{b-a}{n}$), and $\overline{x}_i = \frac{1}{2}(x_{i-1}+x_i)$ is the midpoint of $[x_{i-1}, x_i]$ for $i = 1, \cdots, n$. Therefore,
$$M_n = \Delta x \cdot \sum_{i=1}^{n} f(\overline{x}_i),$$
$$T_n = \frac{1}{2} \cdot \Delta x \cdot \left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right],$$
Then,
$$\begin{aligned} & \frac{1}{3}T_n + \frac{2}{3}M_n \\ = & \frac{1}{3}[T_n + 2M_n] \\ = & \frac{1}{3}\left[\frac{1}{2} \cdot \Delta x \left(\cdot f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right) + 2\left(\Delta x \cdot \sum_{i=1}^{n} f(\overline{x}_i) \right)\right] \\ = & \frac{1}{3} \cdot \left(\frac{\Delta x}{2}\right) \left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) + 4\sum_{i=1}^{n} f(\overline{x}_i) \right] \\ = & S_{2n} \end{aligned}$$