[6.5] Average Value of a Function

2. $f(x) = \sin 4x, [-\pi, \pi]$

Sol. $f_{ave} = \frac{1}{\pi-(-\pi)} \int_{-\pi}^{\pi} \sin 4x dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin 4x \frac{1}{4} d4x = \frac{1}{8\pi}[-\cos 4x]_{-\pi}^{\pi} = 0.$
Remark. $\sin 4x$ is an odd function, so you can also attain the answer without integrating it.

5. $f(t) = te^{-t^2}, [0, 5]$

Sol. $f_{ave} = \frac{1}{5-0} \int_0^5 te^{-t^2} dt = \frac{1}{5} \int_0^5 e^{-t^2} \frac{1}{2} dt^2 = \frac{1}{10} \int_0^5 e^{-t^2} dt^2 = \frac{1}{10}[-e^{-t^2}]_0^5 = \frac{1}{10}[-e^{-25} + 1].$

7. $h(x) = \cos^4 x \sin x, [0, \pi]$

Sol. $h_{ave} = \frac{1}{\pi-0} \int_0^{\pi} \cos^4 x \sin x dx = \frac{1}{\pi} \int_0^{\pi} - \cos^4 x d \cos x = \frac{1}{\pi}[-\frac{1}{5} \cos^5 x]_0^{\pi} = \frac{1}{\pi}(\frac{1}{5} + \frac{1}{5}) = \frac{2}{5\pi}.$

9. $f(x) = (x - 3)^2, [2, 5]$

Sol.
(a) $f_{ave} = \frac{1}{5-2} \int_2^5 (x - 3)^2 dx = \frac{1}{3} \int_2^5 (x - 3)^2 d(x - 3) = \frac{1}{3}[\frac{1}{3}(x - 3)^3]_2^5 = \frac{1}{3}(\frac{1}{3}2^3 - \frac{1}{3}(-1)^3) = 1.$
(b) $(c - 3)^2 = 1 \Rightarrow c = 2, 4.$
(c) See the graph in Appendix I, page A93.

13.

Sol. Let $g(x) = \int_1^x f(t)dt$. Since $g(1) = 0, g(3) = 8$ and g is differentiable in $(1, 3)$ and continuous on $[1, 3]$, by mean value theorem, there is a c such that $f(c) = g'(c) = \frac{g(3)-g(1)}{3-1} = 4.$

14.

Sol. $3 = f_{ave} = \frac{1}{b-0} \int_0^b 2 + 6x - 3x^2 dx = \frac{1}{b}[2x + 3x^2 - x^3]_0^b = \frac{1}{b}(2b + 3b^2 - b^3) = 2 + 3b - b^2$ implies that $b^2 - 3b + 1 = 0$, hence $b = \frac{3\pm\sqrt{5}}{2}$.