Solutions of 4.7

8. We need to maximize P for $I \geq 0$. $P(I) = \frac{100I}{I^2+I+4} \Rightarrow$

$$P'(I) = \frac{(I^2+I+4)(100) - 100I(2I+1)}{(I^2+I+4)^2} = \frac{-100(I^2-4)}{(I^2+I+4)^2}, \text{ so}$$

$$P'(I) > 0 \text{ for } 0 < I < 2 \text{ and } P'(I) < 0 \text{ for } I > 2.$$

Thus, P has an absolute maximum $P(2) = 20$ at $I = 2$.

10. Let x denote the length of the side of the square being cut out. Let y denote the length of the base. Then we have

$$\text{Volume } V = y \cdot y \cdot x = y^2 x. \tag{1}$$

The length of the cardboard $= 3 \Rightarrow$

$$x + y + x = 3 \Rightarrow y = 3 - 2x. \tag{2}$$

Combine (1) and (2), we have $V = V(x) = x(3-2x)^2$. So

$$V'(x) = (3-2x)^2 + x \cdot 2(3-2x)(-2) = (3-2x)(3-6x).$$

So the critical numbers are $x = \frac{3}{2}$ and $x = \frac{1}{2}$.
Now $0 \leq x \leq \frac{3}{2}$ and $V(0) = V(\frac{3}{2}) = 0$, so the maximum is
$V(\frac{1}{2}) = (\frac{1}{2})(2)^2 = 2$ (m^3).

18. Given a point (x, y) on the line $6x + y = 9$. The distance between (x, y) and $(-3, 1)$ is

$$[(x+3)^2 + (y-1)^2]^{\frac{1}{2}} = [(x+3)^2 + (9-6x-1)^2]^{\frac{1}{2}} = (37x^2 - 90x + 73)^{\frac{1}{2}}.$$

Note that minimizing a positive function is equivalent to minimizing the square of the function, so we need to minimize

$$D(x) = (37x^2 - 90x + 73).$$

Now $D'(x) = 74x - 90$, so the critical number is $x = \frac{45}{37}$, and $y = 9 - 6 \cdot \frac{45}{37} = \frac{63}{37}$.
The point on the line that is closest to $(-3, 1)$ is $(\frac{45}{37}, \frac{63}{37})$.

22. Let (x, y), $x, y > 0$, be a vertex of a rectangle inscribed in the ellipse, then the area A of the ellipse is $A = 2x \cdot 2y = 4xy$. Now since (x, y) is on the ellipse, we have $y = \frac{b}{a}\sqrt{a^2 - x^2}$, so we have

$$A = A(x) = \frac{4b}{a} x \sqrt{a^2 - x^2}.$$

So

$$A'(x) = \frac{4b}{a}\left[\sqrt{a^2-x^2} + x\frac{-2x}{2\sqrt{a^2-x^2}}\right] = \frac{a^2-2x^2}{\sqrt{a^2-x^2}}\,.$$

So the maximum of $A(x)$ is

$$A(\frac{a}{\sqrt{2}}) = 2ab.$$

28. Let x be the base radius of a right cylinder inscribed in the cone, and y the height of the cylinder. By similarity of triangles, we see that

$$\frac{y}{r-x} = \frac{h}{r}\,,$$

and so $y = h - \frac{hx}{r}$. The volume V of the cylinder is

$$V = V(x) = (x^2\pi)(h - \frac{hx}{r})\,.$$

Now

$$V'(x) = (2\pi x)(h - \frac{hx}{r}) + (\pi x^2)(-\frac{h}{r}) = h\pi x(2 - \frac{3x}{r})\,.$$

So the maximum of V is $V(\frac{2r}{3}) = \frac{4}{27}\pi r^2 h$.

44. Let h be the time (measured in hour) passing from the leave of the boat traveling south. In this problem we of course consider only those $0 \le h \le 1$. Now by assumption, the boat traveling south is $20h$ long from the dock, and the boat traveling east is $15(1-h)$ long from the dock. By Pythagorean theorem, the distance d between the two boats is

$$d = d(h) = (20h)^2 + (15(1-h))^2,$$

and then we have

$$d'(h) = 2\cdot 20h\cdot 20 + 2\cdot 15(1-h)(-15) = 1250h - 450.$$

So the maximum occurs at $h = 450/1250 = 9/25$. Now $\frac{9}{25}$hr. = 21min. 36sec., so the boats are closest to each other at 2:21:36 PM.

50. The line with slope m ($m < 0$) through $(3,5)$ has equation $y - 5 = m(x-3)$. The y-intercept is $5 - 3m$ and the x-intercept is $3 - \frac{5}{m}$. So the triangle has area

$$A(m) = \frac{1}{2}(5-3m)(3-\frac{5}{m}) = \frac{1}{2}(30 - \frac{25}{m} - 9m).$$

Now

$$A'(m) = \frac{1}{2}(\frac{25}{m^2} - 9),$$

so $A(m)$ has minimum when $m = -\frac{5}{3}$. That is, when the equation of the line is

$$y - 5 = -\frac{5}{3}(x - 3).$$

61. By Pythagorean theorem, we have

$$\begin{aligned} L = L(x) &= x + \sqrt{2^2 + (5 - x)^2} + \sqrt{3^2 + (5 - x)^2} \\ &= x + \sqrt{x^2 - 10x + 29} + \sqrt{x^2 - 10x + 34}. \end{aligned}$$

So

$$L'(x) = 1 + \frac{x - 5}{\sqrt{x^2 - 10x + 29}} + \frac{x - 5}{\sqrt{x^2 - 10x + 34}}.$$

Use a computer and apply any method (e.g. Newton's method) to approximate the solution of $L'(x) = 0$. A good estimate is $x = 3.59$, and the minimum of $L(x)$ is approximately 9.35(m).

64. Let $a = |PQ|$ and $b = |ST|$. We minimize $f(\theta_1) = |PR| + |RS| = a \csc\theta_1 + b \csc\theta_2$. Differentiating with respect to θ_1, and setting $\frac{df}{d\theta_1}$ equal to 0, we get

$$\frac{df}{d\theta_1} = 0 = -a \csc\theta_1 \cot\theta_1 - b \csc\theta_2 \cot\theta_2 \frac{d\theta_2}{d\theta_1}.$$

So we need to find an expression for $\frac{d\theta_2}{d\theta_1}$. We can do this by observing that $|QT| =$ constant $= a \cot\theta_1 + b \cot\theta_2$. Differentiating this equation implicitly with respect to θ_1, we get $-a \csc^2\theta_1 - b \csc^2\theta_2 \frac{d\theta_2}{d\theta_1} = 0$. That is, $\frac{d\theta_2}{d\theta_1} = -\frac{a \csc^2\theta_1}{b \csc^2\theta_2}$. Substitute this into the expression for $\frac{df}{d\theta_1}$ to get $\cos\theta_1 = \cos\theta_2$. Since θ_1 and θ_2 are both acute, we have $\theta_1 = \theta_2$.

66. The length of the pipe cannot exceed $\frac{3}{\sin\theta} + \frac{2}{\cos\theta}$ for $0 < \theta < \frac{\pi}{2}$. So the longest length of the pipe is the minimum of the function $f(\theta) = \frac{3}{\sin\theta} + \frac{2}{\cos\theta}$, $0 < \theta < \frac{\pi}{2}$. Now

$$f'(\theta) = 0 \Leftrightarrow \tan^3\theta = \frac{3}{2}. \qquad (3)$$

Direct calculation shows that (1) implies

$$\sec\theta = [(\frac{3}{2})^{\frac{2}{3}} + 1]^{\frac{1}{2}} \text{ and } \csc\theta = [(\frac{2}{3})^{\frac{2}{3}} + 1]^{\frac{1}{2}}.$$

So the longest length of the pipe is $3[(\frac{2}{3})^{\frac{2}{3}} + 1]^{\frac{1}{2}} + 2[(\frac{3}{2})^{\frac{2}{3}} + 1]^{\frac{1}{2}}$ (m).

73. (a) Let $x = |BC|$, then the distance between the island and C is $\sqrt{x^2 + 25}$. We need to minimize the function $f(x) = 1.4\sqrt{x^2 + 25} + (13 - x)$, $0 \le x \le 13$. Direct calculation shows that $f'(x) = 0 \Leftrightarrow x = \frac{25\sqrt{6}}{12}$. So the point C between B and D that minimizes the energy expended in returning to the nest is $x = \frac{25\sqrt{6}}{12}$ km long from B.

(b) It amounts to replace the coefficient 1.4 of the function f in (a) by $\frac{W}{L}$. Then we have

$$f'(x) = \frac{W}{L}\frac{x}{\sqrt{x^2+25}} - 1.$$

So

$$f'(x) = 0 \Leftrightarrow \frac{W}{L} = \sqrt{\frac{25}{x^2}+1} \Leftrightarrow x = \frac{5}{\sqrt{(\frac{W}{L})^2-1}}.$$

Thus $\frac{W}{L}$ large implies that the distance between B and C is small, and $\frac{W}{L}$ small implies the distance being large.

(c) It's easy to see that $f' \leq 0$ for $x \in [0, 5/\sqrt{(\frac{W}{L})^2 - 1}]$. So f is decreasing in this interval, with minimum at the critical point $x = 5/\sqrt{(\frac{W}{L})^2 - 1}$. But this conclusion is valid only when $5/\sqrt{(\frac{W}{L})^2 - 1} \leq 13$. In fact, if $\frac{W}{L} \leq \frac{\sqrt{194}}{13}$, we have $5/\sqrt{(\frac{W}{L})^2 - 1} \geq 13$. In this case, $f(13)$ is always the minimum of f, and hence the bird should fly directly to its nesting area for minimize the energy expended.

On the other hand, there is no finite value for $\frac{W}{L}$ corresponding to the choice of flying first to B and then flying from B to D. All that we can say is that if $\frac{W}{L}$ is very large, the point C will be very close to B.

(d) The observation infers that the critical point of f is $x = 4$. So from (b), we have

$$\frac{W}{L} = \sqrt{\frac{25}{4^2}+1} = \frac{\sqrt{41}}{4} \approx 1.6.$$