Section 4.4

8. $\lim_{x\to 1}\frac{x^a-1}{x^b-1}\overset{L'hospital}{=}\lim_{x\to 1}\frac{ax^{a-1}}{bx^{b-1}}=\frac{a}{b}$

22. $\lim_{x\to 0}\frac{e^x-1-x-\frac{1}{2}x^2}{x^3}=?$

Since $e^x=1+x+\frac{1}{2}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\ldots,$

$$\lim_{x\to 0}\frac{e^x-1-x-\frac{1}{2}x^2}{x^3}=\lim_{x\to 0}\frac{\frac{1}{3!}x^3+\frac{1}{4!}x^4+\ldots}{x^3}=\frac{1}{6}$$

29. $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2(\frac{x}{2})}{4(\frac{x}{2})^2}=\frac{1}{2}\lim_{x\to 0}\frac{\sin^2(\frac{x}{2})}{(\frac{x}{2})^2}=\frac{1}{2}$

49. $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\lim_{x\to\infty}x\left(\sqrt{1+\frac{1}{x}}-1\right)=\lim_{x\to\infty}\frac{\left(\sqrt{1+\frac{1}{x}}-1\right)}{\frac{1}{x}}$

$$(\frac{0}{0}\Rightarrow L'hospital)=\lim_{x\to\infty}\frac{\frac{1}{2}(1+\frac{1}{x})^{-\frac{1}{2}}\cdot(-x^{-2})}{-x^{-2}}=\frac{1}{2}$$

59. $\lim_{x\to\infty}x^{1/x}=\lim_{x\to 0}e^{\frac{1}{x}\ln x}=e^{\lim_{x\to 0}\frac{1}{x}\ln x}$

$$\lim_{x\to\infty}\frac{1}{x}\ln x=\lim_{x\to\infty}\frac{\ln x}{x}\overset{(\frac{\infty}{\infty}\Rightarrow L'hospital)}{=}\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0$$

So $\lim_{x\to\infty}x^{1/x}=e^0=1$

70. Prove that

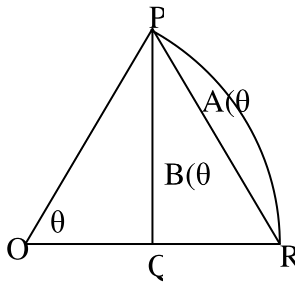
$$\lim_{x\to\infty}\frac{\ln x}{x^p}=0$$

for any number $p>0$. This shows that the logarithmic function approaches ∞ more slowly than any power of x.

Proof: It is the form of $\frac{\infty}{\infty}$, using L'hospital's Rule

$$\lim_{x\to\infty}\frac{\ln x}{x^p}=\lim_{x\to\infty}\frac{x^{-1}}{px^{p-1}}=\lim_{x\to\infty}\frac{1}{p}x^{-p}=0.$$

78. The figure shows a sector of a circle with central angle θ. Let A(θ) be the area of the segment between the chord PR and the arc PR. Let B(θ) be the area of the triangle POR. Find $\lim_{\theta\to0^+} A(\theta)/B(\theta)$.



Without loss of generality, let the radius be 1.

$$A(\theta)=\frac{1}{2}r^2\theta-\frac{1}{2}r\cdot r\sin\theta=\frac{1}{2}(\theta-\sin\theta)$$

$$B(\theta)=\frac{1}{2}r\cdot\sin\theta-\frac{1}{2}r\cos\theta\cdot r\sin\theta=\frac{1}{2}\sin\theta(1-\cos\theta)$$

So $$\lim_{\theta\to0^+}\frac{A(\theta)}{B(\theta)}=\lim_{\theta\to0^+}\frac{\frac{1}{2}(\theta-\sin\theta)}{\frac{1}{2}\sin\theta(1-\cos\theta)}=\lim_{\theta\to0^+}\frac{(\theta-\sin\theta)}{\sin\theta-\sin\theta\cos\theta}=\lim_{\theta\to0^+}\frac{(\theta-\sin\theta)}{\sin\theta-\frac{1}{2}\sin2\theta}$$

$$(\frac{0}{0}\text{=>L'hospital})=\lim_{\theta\to0^+}\frac{1-\cos\theta}{\cos\theta-\cos2\theta}$$

$$(\frac{0}{0}\text{=>L'hospital})=\lim_{\theta\to0^+}\frac{\sin\theta}{-\sin\theta+2\sin2\theta}$$

$$(\frac{0}{0}\text{=>L'hospital})=\lim_{\theta\to0^+}\frac{\cos\theta}{-\cos\theta+4\cos2\theta}=\frac{1}{3}$$

80. For what values of *a* and *b* is the following equation true?

$$\lim_{x\to0}\left(\frac{\sin2x}{x^3}+a+\frac{b}{x^2}\right)=0$$

Observe

$$\lim_{x\to0}\frac{\sin2x}{x^3}+\frac{b}{x^2}=\lim_{x\to0}\frac{\sin2x+bx}{x^3}$$

$$(\frac{0}{0}\text{=>L'hospital Rule})=\lim_{x\to0}\frac{2\cos2x+b}{3x^2}$$

which must be a form of $\frac{0}{0}$ since the given limit exists.

So $b=-2$.

$$(\frac{0}{0}\text{=>L'hospital Rule})=\lim_{x\to0}\frac{-4\sin2x}{6x}=\lim_{x\to0}\frac{-8\cos2x}{6}=\frac{-4}{3}$$

Since the given limit = 0, $a=\frac{4}{3}$

83. Let

$$f(x)=\begin{cases} e^{-1/x^2} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

(a) Use the definition of derivative to compute $f'(0)$

(b) Show that f has derivatives of all orders that are defined on R.

(a) $f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to 0}\dfrac{e^{-1/x^2}-0}{x}=\lim\limits_{x\to 0}\dfrac{\frac{1}{x}}{e^{1/x^2}}=\lim\limits_{x\to 0}\dfrac{\frac{1}{x^2}}{e^{1/x^2}}\cdot x$

Since $\lim\limits_{x\to 0}\dfrac{\frac{1}{x^2}}{e^{1/x^2}}=\lim\limits_{t\to\infty}\dfrac{t}{e^t}=0$ and $\lim\limits_{x\to 0}x=0$, we have

$$f'(0)=\lim_{x\to 0}\frac{\frac{1}{x^2}}{e^{1/x^2}}\cdot x=\lim_{x\to 0}\frac{\frac{1}{x^2}}{e^{1/x^2}}\cdot\lim_{x\to 0}x=0$$

(b) 1° $f'(x)=x^{-2}e^{-1/x^2}=x^{-2}f(x)$

2° Suppose that for n, we have $f^{(n)}(x)=f(x)p_n(x)x^{-k_n}$, where $p_n(x)$ is a polynomial, and k_n is a positive integer for each n.

Consider $n+1$,

$$f^{(n+1)}(x)=f'(x)p_n(x)x^{-k_n}+f(x)p_n'(x)x^{-k_n}+f(x)p_n(x)\left(-k_n\cdot x^{-k_n-1}\right)$$

$$=\left(f(x)\cdot x^{-2}\right)p_n(x)x^{-k_n}+f(x)p_n'(x)x^{-k_n}+f(x)p_n(x)\left(-k_n\cdot x^{-k_n-1}\right)$$

$$=f(x)\cdot\left(p_n(x)+p_n'(x)x^2+p_n(x)\left(-k_n x\right)\right)x^{-k_n-2}$$

$$=f(x)\cdot p_{n+1}(x)x^{-k_{n+1}}$$

where $p_{n+1}(x)$ is a polynomial, and k_{n+1} is a positive integer.

3° By induction, the derivative of each order for f exists.