5.
Sol:

$$\begin{aligned} f(x) &= 1 - x^{2/3} \\ &= 1 - \sqrt[3]{x^2} \\ \Rightarrow f(-1) &= 1 - \sqrt[3]{(-1)^2} \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} f(1) &= 1 - \sqrt[3]{1^2} \\ &= 0 \\ &= f(-1) \end{aligned}$$

Consider

$$f'(x) = \frac{-2}{3}x^{\frac{-1}{3}} \quad = \frac{-2}{3\sqrt[3]{x}}$$

We know that there is no such $c \in \mathcal{R}$ satisfying $f'(c) = \frac{-2}{3\sqrt[3]{c}} = 0$. Q.E.D.

11.
Sol: $f(x)$ is a polynomail
$\Rightarrow f(x)$ is continuous on [-1,1] and differentiable on (-1,1).
$\Rightarrow f(x)$ satisfies the hypotheses of the Mean Value Theorem.
$\Rightarrow \exists c \in (-1, 1)$ such that

$$\begin{aligned} f'(c) &= \frac{f(1) - f(-1)}{1 - (-1)} \\ &= \frac{10 - 6}{2} \\ &= 2 \end{aligned}$$

i.e. $f'(c) = 6c + 2 = 2$
$c = 0$ is one (and the only one) choice satisfying the Mean Value Theorem.
Q.E.D.

18.

Sol: Let $f(x) = 2x - 1 - \sin x$

$\Rightarrow f(x)$ is a continuous and differentiable function on $\mathcal{R}$.

$$f(0) = -1 < 0$$
$$f(\pi) = 2\pi - 1 > 0$$

$\Rightarrow$ There exists at least one $a \in (0, \pi)$ satisfying $f(a) = 0$.

Suppose that there exists another root $b \in \mathcal{R}$ of f(x).

i.e. $f(b) = 0$.

Because $f(x) = 2x - 1 - \sin x$ is differentiable on $\mathcal{R}$, by the mean value theorem, assume that $a < b$

$\Rightarrow \exists c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$.

But $f'(x) = 2 - \cos x \geq 1, \forall x \in \mathcal{R}$.

$\Rightarrow f'(x) \neq 0, \forall x \in (a, b)$, contradiction.

so $f(x) = 2x - 1 - \sin x$ has exactly one real root. Q.E.D.

26.

Sol: Let $h(x) = f(x) - g(x)$, $h(x)$ is continuous on $[a, b]$ and differentiable on (a, b) .

$\Rightarrow h(a) = f(a) - g(a) = 0$ and $h'(x) = f'(x) - g'(x) < 0, \forall x \in (a, b)$.

Now, applying the mean value theorem,

$\Rightarrow \exists c \in (a, b)$ such that $f'(c) = \frac{h(b)-h(a)}{b-a}$.

Because $f'(x) < 0, \forall c \in (a, b)$ and $(b - a) > 0$,

$\Rightarrow h(b) - h(a) < 0$,

i.e. $h(b) = f(b) - g(b) < h(a) = 0$

$\Rightarrow f(b) < g(b)$. Q.E.D.

27.

Sol: Let $f(t)=\sqrt{1+t}-(1+\frac{1}{2}t)$, $f(x)$ is continuous on $[0,\infty)$, differentiable on $(0,\infty)$, and $f(0)=0$.

Consider $f'(t)$,

$$f'(t)=\frac{1}{2\sqrt{1+t}}-\frac{1}{2}=\frac{1}{2}(\frac{1}{\sqrt{1+t}}-1)$$

$\forall t>0, \sqrt{1+t}>1$

$\Rightarrow \frac{1}{\sqrt{1+t}}<\frac{1}{1}=1$

$\Rightarrow \frac{1}{\sqrt{1+t}}-1<0$

$\Rightarrow f'(t)=\frac{1}{2}(\frac{1}{\sqrt{1+t}}-1)<0, \forall t>0.$

Now, $\forall x>0$, we can treat it as a fixed number and applying the mean value theorem,

$\Rightarrow \exists c\in(0,x)$ satisfies $f'(c)=\frac{f(x)-f(0)}{x-0}$

Because $f'(t)<0, \forall t>0$

$\Rightarrow f'(c)=\frac{f(x)-f(0)}{x-0}<0$

$\Rightarrow f(x)=\sqrt{1+x}-(1+\frac{1}{2}x)<0, \forall x>0$

i.e. $\sqrt{1+x}<(1+\frac{1}{2}x), \forall x>0$. Q.E.D.

29.

Sol:

(1) If $a=b$,

$\Rightarrow |\sin a-\sin b|=0$, $|a-b|=0$

$\Rightarrow |\sin a-\sin b|\le|a-b|$, the inequality holds.

(2) If $a\ne b$, assume that $b<a$.

Let $f(x)=\sin x$, $f(x)$ is continuous on $\mathcal{R}$ and differentiable on $\mathcal{R}$.

Apply the mean value theorem on $f(x)$,

$\Rightarrow \exists c\in(b,a)$ satisfies $f'(c)=\frac{f(a)-f(b)}{a-b}$.

Because $f'(x)=\cos x$, $|f'(x)|=|\cos x|\le 1, \forall x\in\mathcal{R}$

$\Rightarrow |\frac{f(a)-f(b)}{a-b}| = |f'(c)| \leq 1$
$\Rightarrow |f(a) - f(b)| \leq |a - b|$
i.e. $|\sin a - \sin b| \leq |a - b|$, the inequality holds.

(3) If $a < b$, similarly, we can prove that $|\sin b - \sin a| \leq |b - a|$
i.e. $|\sin a - \sin b| \leq |a - b|$, the inequality also holds. Q.E.D.