49. Use the *Closed Interval Method*. Solve $f'(x) = 0$, where $f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$. We get 2 roots $x = 2$ and $x = -1$. Notice that both $2, -1 \in [-2, 3]$. After some calculation, we have $f(2) = -19$, $f(-1) = 8$. Now check boundary points. $f(-2) = -3$, $f(3) = -8$. We conclude that in interval $[-2, 3]$, $f(x)$ attends its absolute maximum at $x = -1$ and its absolute minimum at $x = 2$. Its maximum and minimum values are 8 and -19 respectively.

53.
$$f'(x) = \frac{(x)'(x^2+1) - x(x^2+1)'}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$
Solving $f'(x) = 0$ gives $x = 1$ or $x = -1$. But $-1 \notin [0, 2]$, we only consider the critical point $x = 1$. Compute $f(1) = 1/2$. For the boundaries, $f(0) = 0$ and $f(2) = 2/5$. So $f(x)$ attends its absolute maximum and absolute minimum at $x = 1$ and $x = 0$, respectively. The absolute maximum and absolute minimum are $1/2$ and 0.

56. As before, we compute and solve for $f'(t) = 0$, where $f(t) = \sqrt[3]{t}(8-t)$.
$$f'(t) = \frac{1}{3}t^{-2/3}(8-t) + t^{1/3}(-1) = \frac{8-4t}{3t^{2/3}} = \frac{8-4t}{3\sqrt[3]{t^2}}$$
Letting $f'(t) = 0$ gives $t = 2$ and $f(2) = 6\sqrt[3]{2}$. Consider boundaries $t = 0$ and $t = 8$. $f(t)$ has values 0 and any one of both. We conclude that $f(t)$ has absolute maximum at $t = 2$ with value $6\sqrt[3]{2}$ and has absolute minimum at $t = 0$, $t = 8$ with values 0.

60. $f'(x) = 1 - 1/x$. $f'(x)$ only has value zero at $x = 1$. $f(1) = 1$. Now check the boundaries. $f(1/2) = 1/2 + \ln 2$ and $f(2) = 2 - \ln 2$. $\ln 2 \approx 0.6931$. After some simple comparison, we conclude $f(1) = 1$ is the minimum and $f(2) = 1.3069$ is the maximum.

62. $f'(x) = -e^{-x} + 2e^{-2x}$. $f'(x) = 0$ if and only if $e^{-x} = 1/2$. This equation indeed has one solution since e^{-x} is one-to-one. Denote this solution by x_0. We have $f(x_0) = 1/4$. Comparing the boundaries, we have $f(0) = 0$ and $f(1) = e^{-1} - e^{-2x}$. Using $e \approx 2.71828$, we can get $f(1) \approx 0.2325$. So $f(x_0) = 1/4$ is the maximum and $f(0) = 0$ is the minimum.

63. A simple observation, $f(x) > 0$ for all $x \in (0, 1)$, and $f(x) = 0$ as $x = 0$ or $x = 1$. $f(x)$ is countinuous on $[0, 1]$ since a, $b > 0$. By *Extreme Value Theorem*, $f(x)$ has a maximum on some $c \in [0, 1]$ and we know that c is neither 0 nor 1. Using *Power Rule*,
$$f'(x) = ax^{a-1}(1-x)^b - bx^a(1-x)^{b-1} = x^{a-1}(1-x)^{b-1}\left(a(1-x) - bx\right)$$
$f'(x) = 0$ may have zeros at $x = 0$, $x = 1$, or $x = a/(a+b)$. (We say "may" since this depends on whether a or b is equal to or smaller than 1.) However we can exclude the case of $x = 1$ or $x = 0$ here because even if 0 or 1 is a root of $f'(x)$,

they are impossible to be maximums. So we know the maximum must occur at $x = a/(a+b)$ and

$$f\left(\frac{a}{a+b}\right) = \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b$$

67. (a) The graph of $f(x)$ is given below.

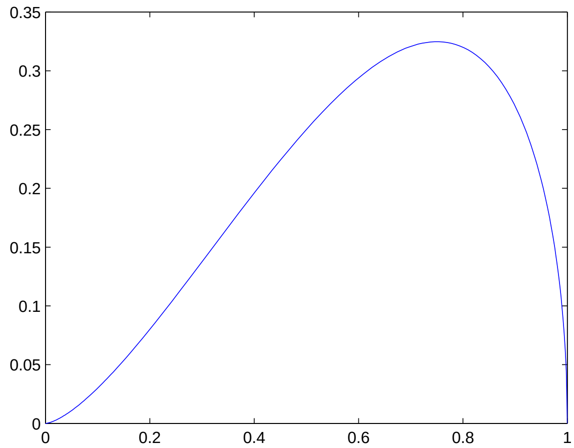

Figure 1: $f(x) = x\sqrt{x - x^2}$

Notice that $f(x)$ has definition only when the value in the square root is nonnegative, i.e., $x - x^2 \geq 0$. $x - x^2 \geq 0$ if and only if $0 \leq x \leq 1$. The maximum and minimum nearly are 0.32 and 0 by using graph.

(b) Using *Calculus*.

$$f'(x) = \sqrt{x - x^2} + \frac{x(1-2x)}{2\sqrt{x - x^2}} = \frac{3x - 4x^2}{2\sqrt{x - x^2}}$$

$f'(x) = 0$ if and only if $x = 3/4$ and

$$f\left(\frac{3}{4}\right) = \frac{3}{16}\sqrt{3}$$

Since $f(x)$ has value zero both at $x = 1$ and $x = 0$. We know that $f(x)$ has a maximum value $3\sqrt{3}/16$ at $x = 3/4$ and minimum 0 at both $x = 0$ and $x = 1$.

78. (a) In *Calculus* class, we only consider real functions. Let $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, $d \in \mathbb{R}$. $f'(x) = 3ax^2 + 2bx + c$. Put $D = (2b)^2 - 4(3a)(c)$. $f(x)$ has 2 distinct real roots if $D > 0$, has a repeated root if $D = 0$, and has no real roots if $D < 0$.

As an example of 2 critical points, let $a = 1/3$, $b = -1$, $c = -8$, $d = 0$. $f(x) = (1/3)x^3 - x^2 - 8x$. Then $f'(x) = x^2 - 2x - 8$. $D = (-2)^2 - 4(1/3)(-8) > 0$, so $f'(x)$ has 2 distinct roots. We illustrate this by graph below.

As an example for exactly one critical point, let $a = 1$, $b = 0$, $c = 0$, $d = 0$. $f(x) = x^3$, and $f'(x) = 3x^2$ has only one root at $x = 0$. We illustrate $f(x)$ below.

To complete all cases, consider $a = 1$, $b = 0$, $c = 1$, $d = 0$. $f(x) = x^3 + x$ and $f'(x) = 3x^2 + 1$ has no real roots. So $f(x)$ has no critical points for all x. We draw this function below, too.

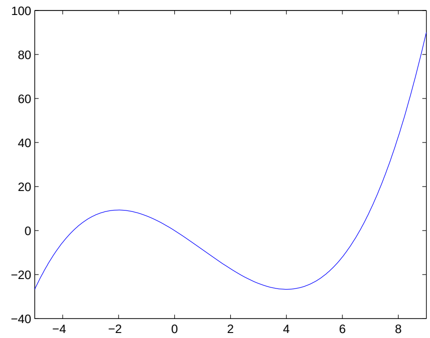

Figure 2: $f(x) = (1/3)x^3 - x^2 - 8x$

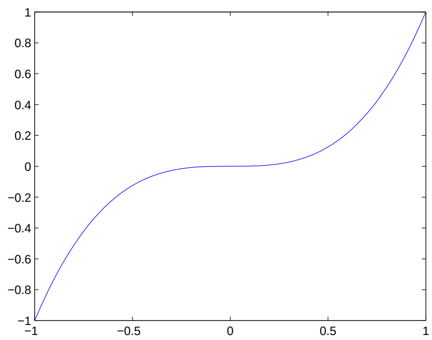

Figure 3: $f(x) = x^3$

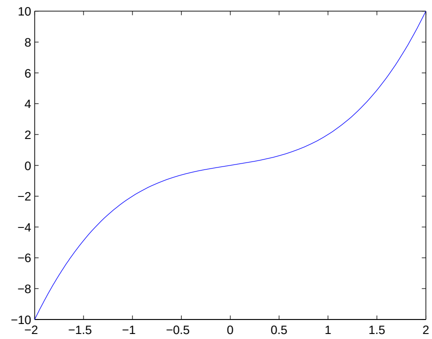

Figure 4: $f(x) = x^3 + x$

(b) By (a), a cube function can have 2, 1, or 0 critical points.