SECTION 3.5

5. Differentiating the equation implicitly with respect to x,

$$3x^2 + 3y^2 \frac{dy}{dx} = 0.$$

So

$$\frac{dy}{dx} = -\frac{x^2}{y^2}.$$

25. Differentiating the equation implicitly with respect to x,

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(2x+y)}{x+2y}.$$

So

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \left.\frac{-(2x+y)}{x+2y}\right|_{(1,1)} = \frac{-3}{3} = -1.$$

Hence the tangent line is

$$(y-1) = -(x-1) \Leftrightarrow y = -x+2.$$

26. Differentiating the equation implicitly with respect to x,

$$2x + 2y + 2x\frac{dy}{dx} - 2y\frac{dy}{dx} + 1 = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(2x+2y+1)}{2x-2y}.$$

So

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \left.\frac{-(2x+2y+1)}{2x-2y}\right|_{(1,2)} = \frac{7}{2}.$$

Hence the tangent line is

$$(y-2) = \frac{7}{2}(x-1) \Leftrightarrow y = \frac{7}{2}x - \frac{3}{2}.$$

27. Differentiating the equation implicitly with respect to x,

$$2x + 2y\frac{dy}{dx} = 2(2x^2+2y^2-x)(4x+4y\frac{dy}{dx}-1).$$

Take $(x,y) = (0,\frac{1}{2})$ into this equation,

$$\left.\frac{dy}{dx}\right|_{(0,\frac{1}{2})} = 1.$$

Hence the tangent line is

$$(y-\frac{1}{2}) = x \Leftrightarrow y = x + \frac{1}{2}.$$

28. Differentiating the equation implicitly with respect to x,

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y^{-\frac{1}{3}}}{x^{-\frac{1}{3}}}.$$

So

$$\left.\frac{dy}{dx}\right|_{(-3\sqrt{3},1)} = \frac{1}{\sqrt{3}}.$$

Hence the tangent line is

$$(y-1) = \frac{1}{\sqrt{3}}(x+3\sqrt{3}) \Leftrightarrow y = \frac{1}{\sqrt{3}}x + 4.$$

29. Differentiating the equation implicitly with respect to x,

$$4(x^2+y^2)(2x+2y\frac{dy}{dx}) = 25(2x-2y\frac{dy}{dx})$$

Take $(x,y) = (3,1)$ into this equation,

$$\left.\frac{dy}{dx}\right|_{(3,1)} = -\frac{9}{13}.$$

Hence the tangent line is

$$(y-1) = -\frac{9}{13}(x-3) \Leftrightarrow y = -\frac{9}{13}x + \frac{40}{13}.$$

30. Differentiating the equation implicitly with respect to x,

$$2y\frac{dy}{dx}(y^2-4) + y^2(2y)\frac{dy}{dx} = 2x(x^2-5) + x^2(2x)$$

Take $(x,y) = (0,-2)$ into this equation,

$$\left.\frac{dy}{dx}\right|_{(0,-2)} = 0.$$

Hence the tangent line is

$$y+2 = 0 \Leftrightarrow y = -2.$$

59. $x^2+y^2 = r^2$ is a circle with center O, and $ax+by=0$ is a line through center O (assume $a^2+b^2 \neq 0$).

$$x^2+y^2 = r^2 \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}.$$

For a point $P_0(x_0,y_0)$,

$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = -\frac{x_0}{y_0},$$

and the slope of the line OP_0 is $\frac{y_0}{x_0}$. Hence, the curves are orthogonal, and the families of curves are orthogonal trajectories of each other.

63. Take $y = 0$ into this equation, $x = \pm\sqrt{3}$.

$$x^2 - xy + y^2 = 3 \Rightarrow 2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{2x - y}{x - 2y}$$

so

$$\left.\frac{dy}{dx}\right|_{(\sqrt{3},0)} = \left.\frac{dy}{dx}\right|_{(-\sqrt{3},0)} = 2.$$

The two tangent lines are

$$y = 2(x - \sqrt{3}), y = 2(x + \sqrt{3})$$

which are parallel.

69. Differentiating the equation implicitly with respect to x,

$$2x + 8y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{4y}.$$

Let h be the height of the lamp and (a, b) be the intersection point of tangent and the ellipse. The tangent line has slope

$$-\frac{h}{8} = -\frac{a}{4b} = \frac{b}{a + 5}$$

By

$$a^2 + 4b^2 = 5,$$

we can compute

$$a = -1, b = 1, h = 2.$$