Section 10.3

Polar Coordinates

66. $r = e^\theta \Rightarrow x = r\cos\theta = e^\theta\cos\theta,\ y = r\sin\theta = e^\theta\sin\theta.$

$\Rightarrow \frac{dy}{d\theta} = e^\theta\sin\theta + e^\theta\cos\theta = e^\theta(\sin\theta+\cos\theta),\ \frac{dx}{d\theta} = e^\theta\cos\theta - e^\theta\sin\theta = e^\theta(\cos\theta - \sin\theta).$

Let $\frac{dy}{d\theta} = 0 \Rightarrow \sin\theta = -\cos\theta \Rightarrow \tan\theta = -1 \Rightarrow \theta = \frac{-1}{4}\pi + n\pi$ (n any integer) $\Rightarrow$ horizontal tangents at $(e^{\pi(n-\frac{1}{4})},\ \pi(n-\frac{1}{4}))$.

Let $\frac{dx}{d\theta} = 0 \Rightarrow \sin\theta = \cos\theta \Rightarrow \tan\theta = 1$

$\Rightarrow \theta = \frac{1}{4}\pi + n\pi$ (n any integer)

$\Rightarrow$ vertical tangents at $(e^{\pi(n+\frac{1}{4})},\ \pi(n+\frac{1}{4}))$.

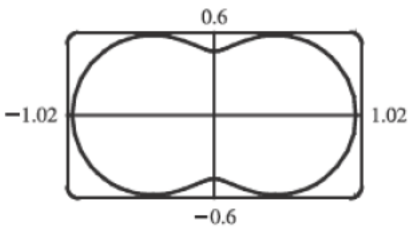
68. By differentiating implicitly, $r^2 = \sin 2\theta \Rightarrow 2r(\frac{dr}{d\theta}) = 2\cos 2\theta \Rightarrow \frac{dr}{d\theta} = \frac{1}{r}\cos 2\theta$.

So, $\frac{dy}{d\theta} = (\frac{dr}{d\theta})\sin\theta + r\cos\theta = \frac{1}{r}\cos 2\theta\sin\theta + r\cos\theta = \frac{1}{r}(\cos 2\theta\sin\theta + r^2\cos\theta)$
$= \frac{1}{r}(\cos 2\theta\sin\theta + \sin 2\theta\cos\theta) = \frac{1}{r}\sin 3\theta$.

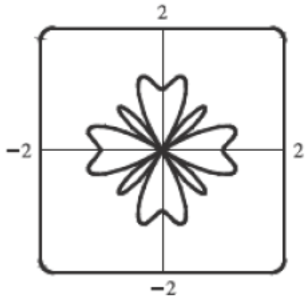
$\frac{dy}{d\theta} = 0 \Rightarrow \sin 3\theta = 0 \Rightarrow \theta = 0,\ \frac{\pi}{3}$ or $\frac{4}{3}\pi$ (restricting θ to the domain of the lemniscate). So, there are horizontal tangents at $(\sqrt[4]{\frac{3}{4}}, \frac{\pi}{3})$, $(\sqrt[4]{\frac{3}{4}}, \frac{4\pi}{3})$ and $(0,0)$. Similarly, $\frac{dx}{d\theta} = \frac{1}{r}\cos 3\theta = 0$ when $\theta = \frac{\pi}{6}$ or $\frac{7\pi}{6}$, so there are vertical tangents at $(\sqrt[4]{\frac{3}{4}}, \frac{\pi}{6})$, $(\sqrt[4]{\frac{3}{4}}, \frac{7\pi}{6})$ [and $(0,0)$].

70. These curves are circles which intersect at the origin and at $(\frac{a}{\sqrt{2}}, \frac{\pi}{4})$. At the origin, the first circle has a horizontal tangent and the second a vertical one, so the tangents are perpandicular here. For the first circle $[r = a\sin\theta]$, $\frac{dy}{d\theta} = a\cos\theta\sin\theta + a\sin\theta\cos\theta = a\sin 2\theta = a$ at $\theta = \frac{\pi}{4}$ and $\frac{dx}{d\theta} = a\cos^2\theta - a\sin^2\theta = a\cos 2\theta = 0$ at $\theta = \frac{\pi}{4}$, so the tangent here is vertical. Similarly, for the second circle $[r = a\cos\theta]$, $\frac{dy}{d\theta} = a\cos 2\theta = 0$ and $\frac{dx}{d\theta} = -a\sin 2\theta = -a$ at $\theta = \frac{\pi}{4}$, so the tangent is horizontal, and again the tangents are perpendicular.

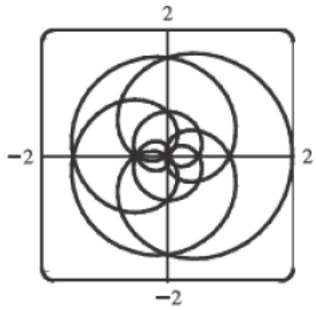
72. $r = \sqrt{1 - 0.8\sin^2\theta}$. The parameter interval is $[0, 2\pi]$.



74. $r = \sin^2(4\theta) + \cos(4\theta)$. The parameter interval is $[0, 2\pi]$.



76. $r = \cos(\frac{\theta}{2}) + \cos(\frac{\theta}{3})$. The parameter interval is $[-6\pi, 6\pi]$.



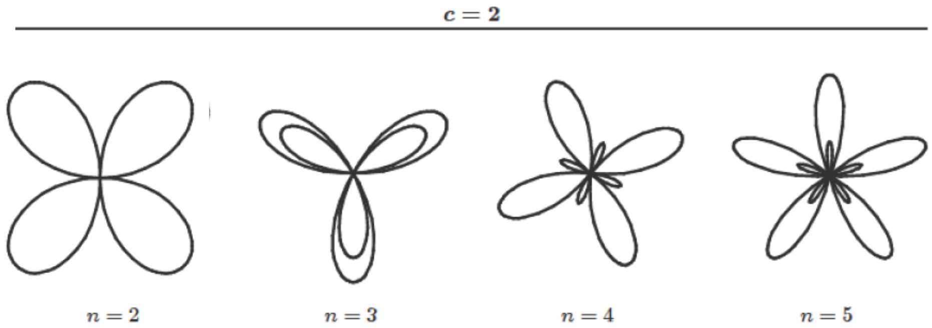
78. From the graph, the highest points seem to have $y \approx 0.77$. To find the exact value, we solve $\frac{dy}{d\theta} = 0$. $y = r\sin\theta = \sin\theta\sin 2\theta \Rightarrow \frac{dy}{d\theta} = 2\sin\theta\cos 2\theta + \cos\theta\sin 2\theta = 2\sin\theta(2\cos^2\theta - 1) + \cos\theta(2\sin\theta\cos\theta) = 2\sin\theta(3\cos^2\theta - 1)$.
In the first quadrant, this is 0 when $\cos\theta = \frac{1}{\sqrt{3}} \Leftrightarrow \sin\theta = \sqrt{\frac{2}{3}} \Leftrightarrow y = 2\sin^2\theta\cos\theta = 2 \cdot \frac{2}{3} \cdot \frac{1}{\sqrt{3}} = \frac{9}{4}\sqrt{3} \approx 0.77$.

79. (a) $r = \sin n\theta$.
From the graphs, it seems that when n is even, the number of loops in the curve (called a rose) is $2n$, and when n is odd, the number of loops is simply n. This is because in the case of n odd, every point on the graph is traversed twice, due to the fact that

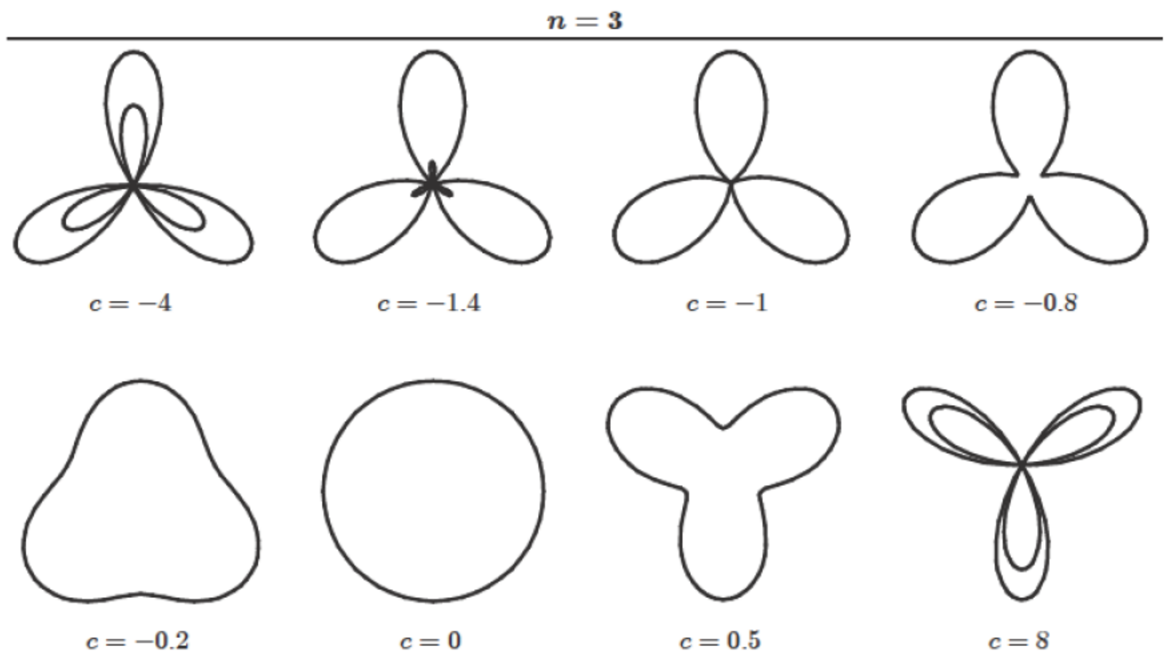
$$r(\theta+\pi) = \sin[n(\theta+\pi)] = \sin n\theta\cos n\pi + \cos n\theta\sin n\pi = \begin{cases} \sin n\theta & \text{if } n \text{ is even,} \\ -\sin n\theta & \text{if } n \text{ is odd,} \end{cases}$$

$n=2$ $n=3$ $n=4$ $n=5$

80. $r = 1 + c \sin n\theta$. We vary n while keeping c constant at 2. As n changes, the curves change in the same way as those in Exercise 79: the number of loops increases. Note that if n is even, the smaller loops are outside the larger ones; if n is odd, there are inside.



Now we vary c while keeping $n = 3$. As c increases toward 0, the entire graph gets smaller (the graphs below are not to scale) and the smaller loops shrink in relation to the large ones. At $c = -1$, the small loops disappear entirely, and for $-1 < c < 1$, the graph is a simple, closed curve (at $c = 0$ it is a circle). As c continues to increase, the same changes are seen, but in reverse order, since $1 + (-c) \sin n\theta = 1 + c \sin n(\theta + \pi)$, so the graph for $c = c_0$ is the same as that for $c = -c_0$, with a rotation through π. As $c \to \infty$, the smaller loops get relatively closer in size to the large ones. Note that the distance between the outermost points of corresponding inner and outer loops is always 2. Maple's animate command (or Mathematica's Animate) is very useful for seeing the changes that occur as c varies.

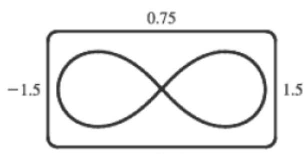


82. Most graphing devices cannot plot implicit polar equations, so we must first find an explicit expression (or expressions) for r in terms of θ, a, and c. We note that the given equation, $r^4 - 2c^2r^2\cos 2\theta + c^4 - a^4 = 0$, is a quadratic in r^2, so we use the quadratic formula and find that

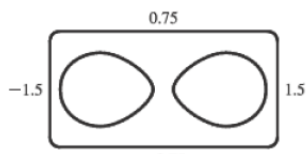
$$r^2 = \frac{2c^2\cos 2\theta \pm \sqrt{4c^4\cos^2 2\theta - 4(c^4 - a^4)}}{2} = c^2\cos 2\theta \pm \sqrt{a^4 - c^4\sin^2 2\theta}$$

so $r = \pm\sqrt{c^2\cos 2\theta \pm \sqrt{a^4 - c^4\sin^2 2\theta}}$. So for each graph, we must plot four curves to be sure of plotting all the points which satisfy the given equation. Note that all functions have period π.

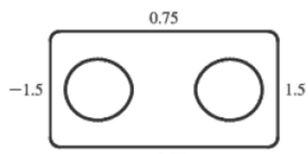
We start with the case $a = c = 1$, and the resulting curve resembles the symbol for infinity. If we let a decrease, the curve splits into two symmetric parts, and as a decrease further, the parts become smaller, further apart, and rounder. If instead we let a increase from 1, the two lobes of the curve join together, and as a increases further they continue to merge, until at $a \approx 1.4$, the graph no longer has dimples, and has an oval shape. As $a \to \infty$, the oval becomes larger and rounder, since the c^2 and c^4 terms lose their significance. Note that the shape of the graph seems to depend only on the ratio c/a, while the size of the graph varies as c and a jointly increase.



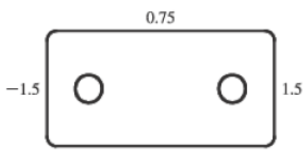
$(a, c) = (1, 1)$



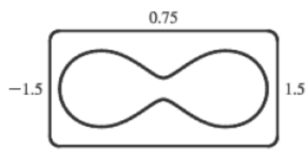
$(a, c) = (0.99, 1)$



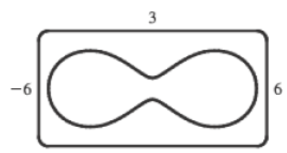
$(a, c) = (0.9, 1)$



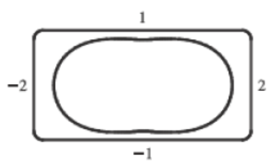
$(a, c) = (0.6, 1)$



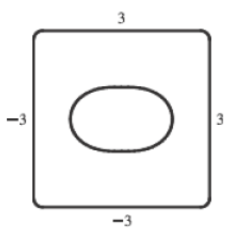
$(a, c) = (1.01, 1)$



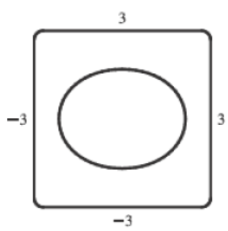
$(a, c) = (4.04, 4)$



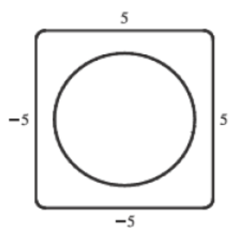
$(a, c) = (1.3, 1)$



$(a, c) = (1.5, 1)$



$(a, c) = (2, 1)$



$(a, c) = (4, 1)$

84. (a)$r = e^\theta \Rightarrow \frac{dr}{d\theta} = e^\theta$, so by Exercise 83, $\tan\psi = \frac{r}{e^\theta} = 1 \Rightarrow \psi = \arctan 1 = \frac{\pi}{4}$. (b)The Cartesian equation of the tangent line at $(1, 0)$ is $y = x - 1$, and that of the tangent line at $(0, e^{\pi/2})$ is $y = e^{\pi/2} - x$. (c)Let a be the tangent of the angle between the tangent and radial lines, that is, $a = \tan\psi$. Then, by Exercise 83, $a = \frac{r}{dr/d\theta} \Rightarrow \frac{dr}{d\theta} = \frac{1}{a}r \Rightarrow r = Ce^{\theta/a}$ [by Theorem 9.4.2]