九十學年度下學期微積分甲 (I) 組01-06班期中考題

(1) 判別下列函數在$(x,y)\to(0,0)$時, 其極限值是否存在.若是存在, 給出算式求其值; 若不存在, 詳述理由為何.
(i)$f(x,y)=\frac{x^2-y^2}{x^2+y^2}$(4分) . (ii)$g(x,y)=\frac{x^3-y^3}{x^2+y^2}$(6分) .

(解)

(i) For $x=0$, $y\neq 0$, $f(x,y)=-1$; while for $x\neq 0$, $y=0$, $f(x,y)=1$. Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exsit.

(ii) For $(x,y)\neq(0,0)$, we have $|g(x,y)|\leq |\frac{x^3}{x^2+y^2}|+|\frac{y^3}{x^2+y^2}|\leq |x|+|y|$.
If $(x,y)\to(0,0)$, $|x|+|y|\to 0$, so by sequeeze principle,
$\lim_{(x,y)\to(0,0)} g(x,y)=0$

(2)求由兩曲面$x^3+3x^2y^2+y^3+4xy-z^2=0$ 與 $x^2+y^2+z^2=11$ 相交出的曲線在點 $(1,1,3)$處的切線參數式. (10分)

解 $\nabla f=(3x^2+6xy^2+4y)i+(6x^2y+3y^2+4x)j-2zk\Rightarrow\nabla f(1,1,3)=13i+13j-6k$; $\nabla g=2xi+2yj+2zk\Rightarrow\nabla g(1,1,3)=2i+2j+6k$; $v=\nabla f\times\nabla g\Rightarrow$

$$v=\left|\begin{array}{ccc} i & j & k\\ 13 & 13 & -6\\ 2 & 2 & 6\end{array}\right|=90i-90j$$

$\Rightarrow$ Tangent line: $x=1+90t$, $y=1-90t$, $z=3$

(3) 設$w=f(x,y)$為可微分函數, 令$x=r\cos\theta$, $y=r\sin\theta$.

(i) 求$\frac{\partial w}{\partial r}$與$\frac{\partial w}{\partial\theta}$.(5分)

(ii) 試証$(\frac{\partial w}{\partial x})^2+(\frac{\partial w}{\partial y})^2=(\frac{\partial w}{\partial r})^2+\frac{1}{r^2}(\frac{\partial w}{\partial\theta})^2$.(5分)

(解)

(i) 由連鎖規則知
$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial w}{\partial x}\cos\theta+\frac{\partial w}{\partial y}\sin\theta$
$\frac{\partial w}{\partial\theta}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial\theta}=-\frac{\partial w}{\partial x}r\sin\theta+\frac{\partial w}{\partial y}r\cos\theta$

(ii) 解上面兩式得
$\frac{\partial w}{\partial x}=\cos\theta\frac{\partial w}{\partial r}-\frac{\sin\theta}{r}\frac{\partial w}{\partial r}$
$\frac{\partial w}{\partial y}=\sin\theta\frac{\partial w}{\partial r}+\frac{\cos\theta}{r}\frac{\partial w}{\partial\theta}$
於是

$$\begin{aligned}(\frac{\partial w}{\partial x})^2+(\frac{\partial w}{\partial y})^2 &= [\cos\theta\frac{\partial w}{\partial r}-\frac{\sin\theta}{r}\frac{\partial w}{\partial r}]^2+[\sin\theta\frac{\partial w}{\partial r}+\frac{\cos\theta}{r}\frac{\partial w}{\partial\theta}]^2\\ &= (\frac{\partial w}{\partial r})^2+\frac{1}{r^2}(\frac{\partial w}{\partial y})^2\end{aligned}$$

(4) 設$f(x,y,z)$在一點P, 沿 $\vec{u}=\vec{i}+\vec{j}-\vec{k}$的方向具有最大的方向導數$2\sqrt{3}$

(i) 求梯度$\nabla f(P)$. (5分)

(ii) 求f在P點, 沿 $\vec{i}+\vec{j}$方向的方向導數. (5分)

(解)

(i) 因梯度∇f的方向是f增加最快的方向, 所以$\nabla f(P)$與$\vec{u}=\vec{i}+\vec{j}-\vec{k}$同向, 即$\nabla f(P)=\alpha\vec{u}=\alpha\vec{i}+\alpha\vec{j}-\alpha\vec{k}$,$\alpha>0$於是$\frac{\partial f}{\partial x}(P)=\alpha,\frac{\partial f}{\partial y}(P)=\alpha,\frac{\partial f}{\partial z}(P)=-\alpha$又已知最大的方向導數爲$2\sqrt{3}$, 亦即 $\sqrt{[\frac{\partial f}{\partial x}(P)]^2+[\frac{\partial f}{\partial y}(P)]^2+[\frac{\partial f}{\partial z}(P)]^2}=2\sqrt{3}$, $\sqrt{3\alpha^2}=2\sqrt{3}$, $\alpha=\pm 2$(負的不合), 所以$\nabla f(P)=2\vec{i}+2\vec{j}-2\vec{k}$

(ii)

$$\begin{aligned}\nabla f(P)\cdot\frac{\vec{i}+\vec{j}}{\sqrt{2}} &= (2\vec{i}+2\vec{j}-2\vec{k})\cdot\frac{\vec{i}+\vec{j}}{\sqrt{2}}\\ &= 2\cdot\frac{1}{\sqrt{2}}+2\cdot\frac{1}{\sqrt{2}}=2\sqrt{2}\end{aligned}$$

(5) 設點 (x,y,z) 在球面 $x^2+y^2+z^2=30$ 上的溫度爲 $T(x,y,z)=4xy+4y^2+2yz$. 試求出該球面上最高溫、最低溫的位置. (10分)

(解) Let $T=f(x,y,z)=4xy+4y^2+2yz$, $g(x,y,z)=x^2+y^2-z^2-30$ by Lagrange Multiplier Method. $\nabla f=\lambda\nabla g$. So, $(4y,4x+8y+2z,2y)=\lambda(2x,2y,2z)$

(I) Suppose $\lambda=0$, so $y=0$, $4x+2z=0$, $z=-2x$. Hence, $x^2+0^2+(-2x)^2=30$, $x^2=6$, $x=\pm\sqrt{6}$. So $x=-\sqrt{6}$, $y=0$, $z=-2\sqrt{6}$, $f(\sqrt{6},0,-2\sqrt{6})=0$; $x=-\sqrt{6}$, $y=0$, $z=2\sqrt{6}$, $f(-\sqrt{6},0,2\sqrt{6})=0$.

(II) Suppose $\lambda\neq 0$, so $2x=4z$, $x=2z$, $y=\lambda z$. If $z=0$, then $x=0$, $y=0$(不合). Hence $z\neq 0$, thus $4x+8y+2z=2\lambda y$, $8z+8\lambda z+2z=2\lambda^2 z$, so $\lambda^2-4\lambda-5=0$, $\lambda=5,-1$.

(i) Suppose $\lambda=5$, $x=2z$, $y=5z$, so $(2z)^2+(5z)^2+z^2=30$, $z^2=1$, $z=\pm 1$. Hence, $x=2$, $y=5$, $z=1$, $f(2,5,1)=150$; $x=-2$, $y=-5$, $z=-1$, $f(-2,-5,-1)=150$

(ii) Suppose $\lambda=-1$, $x=2z$, $y=2z$, so $(2z)^2+(-z)^2+z^2=30$, $z^2=5$, $z=\pm\sqrt{5}$. Hence, $x=2\sqrt{5}$, $y=-\sqrt{5}$, $z=\sqrt{5}$, $f(2\sqrt{5},-\sqrt{5},\sqrt{5})=-30$; $x=-2\sqrt{5}$, $y=\sqrt{5}$, $z=-\sqrt{5}$, $f(-2\sqrt{5},\sqrt{5},-\sqrt{5})=-30$

Therefore, the highest temperature is 150 at $(2,5,1)$ or $(-2,-5,-1)$; the lowest temperature is -30 at $(2\sqrt{5},-\sqrt{5},\sqrt{5})$ or $(-2\sqrt{5},\sqrt{5},-\sqrt{5})$

(6) 試求函數 $f(x,y)=x^2+2xy+2y^2-4x-6y$ 在由 $(0,0)$, $(0,2)$, $(3,0)$此三點所圍成的三角區域上 (含內部及邊界) 所產生的最大値與最小値. (10分)

(解)

I) $f(x,y)=x^2+2xy+2y^2-4x-6y$ $\begin{cases} \frac{\partial f}{\partial x}=2x+2y-4=0 \\ \frac{\partial f}{\partial y}=2x+4y-6=0 \end{cases}$, so, $x=1$, $y=1$.

II) Suppose $x=0$, let $g(y)=f(0,y)=2y^2-6y$. $g'(y)=4y-6=0$. So $y=\frac{3}{2}$. So $x=0$, $y=\frac{3}{2}$

III) Suppose $y=0$, let $h(x)=f(x,0)=x^2-4x$, $h'(x)=2x-4=0$. So $x=2$. So $x=2$, $y=0$.

IV) Suppose $2x+3y-6=0$, so $y=-\frac{2}{3}x+\frac{6}{3}$. Let $u(x)=f(x,-\frac{2}{3}x+\frac{6}{3})=x^2+2x(-\frac{2}{3}x+\frac{6}{3})+2(-\frac{2}{3}x+\frac{6}{3})^2-4x-6(-\frac{2}{3}x+\frac{6}{3})$, so $u'(x)=2x+\frac{-4}{3}\cdot 2x+\frac{12}{3}+2(-\frac{2}{3}x+\frac{6}{3})\cdot\frac{-2}{3}-4+6\cdot\frac{2}{3}=0$, so $x=\frac{6}{5}$, $y=-\frac{2}{3}x+\frac{6}{3}=\frac{6}{5}$
$f(1,1)=-5$, $f(0,\frac{3}{2})=\frac{-9}{2}$, $f(0,0)=0$, $f(2,0)=-4$, $f(3,0)=-3$, $f(\frac{6}{5},\frac{6}{5})=\frac{-24}{5}$, $f(0,2)=-4$.

Hence, the abs. max value of $f(x,y)$ is $f(0,0)=0$; the abs. min value of $f(x,y)$ is $f(1,1)=-5$.

(7) 試求由 $x^2=z$, $x^2=4-z$, $y=0$, $z+2y=4$ 諸曲面所圍成立體的體積.
(10分)

(解) 解 $\begin{cases} x^2=z, \\ x^2=4-z \end{cases}$ 得$x=\pm\sqrt{2}$, $z=2$, 故所求體積爲

$$\begin{aligned}
& \int_{-\sqrt{2}}^{\sqrt{2}}\int_{x^2}^{4-x^2}(2-\frac{z}{2})dzdx \\
= & \int_{-\sqrt{2}}^{\sqrt{2}}[2z-\frac{z^2}{4}]_{x^2}^{4-x^2}dx \\
= & \int_{-\sqrt{2}}^{\sqrt{2}}(4-2x^2)dx \\
= & 2\int_{0}^{\sqrt{2}}(4-2x^2)dx \\
= & 2[4x-\frac{2x^3}{3}]_0^{\sqrt{2}}=\frac{16\sqrt{2}}{3}.
\end{aligned}$$

(8) 設a爲一正數, 試求圓柱體$x^2+y^2\le ay$與球體$x^2+y^2+z^2\le a^2$共同部分的體積. (10分)

(解) $D=\{(x,y)\in R^2|x^2+y^2\le ay\}$

$$\begin{aligned} V &= 4\int\int_D \sqrt{a^2-x^2-y^2}dA = 4\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta} r\sqrt{a^2-r^2}drd\theta \\ &= 4\int_0^{\frac{\pi}{2}}[-\frac{1}{3}(a^2-r^2)^{\frac{3}{2}}]_0^{a\sin\theta}d\theta = \frac{4a^3}{3}\int_0^{\frac{\pi}{2}}(1-\cos^3\theta)d\theta \\ &= \frac{4a^3}{3}[\theta-(\sin\theta-\frac{1}{3}\sin^3\theta)]_0^{\frac{\pi}{2}} = \frac{4a^3}{3}(\frac{\pi}{2}-\frac{2}{3}) \end{aligned}$$

(9) 求心臟線 $r=1+\cos\theta$ 其內部區域的形心位置. (10分)

(解) Let $M_y=\int\int_\Omega xdxdy$, $M_x=\int\int_\Omega ydxdy=0$ by symmetry $M=\int\int_\Omega dxdy$. We have $M=\int_0^{2\pi}\int_0^{1+\cos\theta}rdrd\theta=\int_0^{2\pi}\frac{(1+\cos\theta)^2}{2}d\theta=\frac{3\pi}{2}$ $M_y=\int_0^{2\pi}\int_0^{1+\cos\theta}r\cos\theta rdrd\theta=\int_0^{2\pi}\frac{(1+\cos\theta)^3}{3}\cos\theta d\theta=\int_0^{2\pi}(\cos^2\theta+\frac{\cos^4\theta}{3})d\theta=\pi+\frac{\pi}{4}=\frac{5\pi}{4}$. 所以 $\bar{x}=\frac{M_y}{M}=\frac{5}{6}$, $\bar{y}=\frac{M_x}{M}=0$

(10) 計算 $\int_0^1\int_{\sqrt[3]{z}}^1\int_0^{\ln 3}\frac{\pi e^{2x}\sin\pi y^2}{y^2}dxdydz$. (10分)

(解)

$$\begin{aligned} \text{Ans} &= \int_0^1\int_{\sqrt[3]{z}}^1\int_0^{\ln 3}\frac{\pi e^{2x}\sin\pi y^2}{y^2}dxdydz \\ &= \int_0^1\int_{\sqrt[3]{z}}^1[\frac{\pi e^{2x}\sin\pi y^2}{2y^2}]_0^{\ln 3}dydz \\ &= \int_0^1\int_{\sqrt[3]{z}}^1\frac{4\pi\sin\pi y^2}{y^2}dydz \quad \left\{\begin{array}{l} 0\le z\le 1 \\ \sqrt[3]{z}\le y\le 1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} 0\le y\le 1 \\ 0\le z\le y^3 \end{array}\right. \\ &= \int_0^1\int_0^{y^3}\frac{4\pi\sin\pi y^2}{y^2}dzdy \\ &= \int_0^1 4\pi y\sin\pi y^2dy = [-2\cos\pi y^2]_0^1 = 4. \end{aligned}$$