(1) 求 $\int\int_E \sin(x+y)\cos(2x-y)dA$, 此處 E 是由 $y=2x-1$, $y=2x+3$, $y=-x$ 和 $y=-x+1$所圍成的區域. (提示: 可用變數變換 $u=2x-y$, $v=x+y$) (12分)

Sol. Let $u=2x-y$, $v=x+y$, then

$$\begin{array}{lcl} y=2x-1 & \Rightarrow & u=1 \\ y=2x+3 & \Rightarrow & u=-3 \\ y=-x & \Rightarrow & v=0 \\ y=-x+1 & \Rightarrow & v=1 \end{array}$$

$$\int\int_{E_{xy}} \sin(x+y)\cos(2x-y)dA = \int\int_{E_{uv}} \sin v \cos u \cdot |\frac{\partial(x,y)}{\partial(u,v)}|dudv$$

From $u=2x-y$, $v=x+y$, we have $3x=u+v \Rightarrow x=\frac{1}{3}(u+v)$, $y=\frac{1}{3}(-u+2v)$. Thus, $|\frac{\partial(x,y)}{\partial(u,v)}| = \left\| \begin{array}{cc} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{array} \right\| = \left\| \begin{array}{cc} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{array} \right\| = \frac{1}{3}$. The integral becomes

$$\frac{1}{3}\int_0^1\int_{-3}^1 \sin v \cos u dudv = \frac{1}{3}(\sin 1 + \sin 3)(1-\cos 1)$$

(2) 在力場 $\vec{F}(x,y) = -y^2\vec{i} + x^2\vec{j}$ 的作用下, 沿著圓形路徑 $x^2+y^2=2$ 以逆時針方向由點 $(\sqrt{2},0)$ 至點 $(-\sqrt{2},0)$ 移動一物體所需做的功爲多少? (10分)

Sol. $\int \vec{F}\cdot d\vec{r} = \int_0^\pi (-2\sin^2\theta, 2\cos^2\theta)\cdot(-\sqrt{2}\sin\theta, \sqrt{2}\cos\theta)d\theta = 2\sqrt{2}\int_0^\pi (\cos^3\theta + \sin^3\theta)d\theta$

$$\begin{aligned} \int_0^\pi \cos^3\theta d\theta &= \int_0^\pi (1-\sin^2\theta)d\sin\theta = \sin\theta - \frac{\sin^3\theta}{3}|_0^\pi \\ &= 0 \end{aligned}$$

$$\begin{aligned} \int_0^\pi \sin^3\theta d\theta &= -\int_0^\pi (1-\cos^2\theta)d\cos\theta = -(\cos\theta - \frac{\cos^3\theta}{3})|_0^\pi \\ &= -(-1+\frac{1}{3}) + (1-\frac{1}{3}) \\ &= 2(1-\frac{1}{3}) = \frac{4}{3} \end{aligned}$$

$\Rightarrow \int \vec{F}\cdot d\vec{r} = 2\sqrt{2}\cdot\frac{4}{3} = \frac{8\sqrt{2}}{3}$

(3) 試求線積分

$$\int_{C_1 \bigcup C_2} xydx + yzdy + zxdz,$$

此處 C_1 表示由點 $(0,0,0)$ 至點 $(1,1,0)$ 的有向線段, 而 C_2 表示由點 $(1,1,0)$ 至點 $(1,1,1)$ 的有向線段. (12 分)

Sol.

1. $C_1 = \{ti + tj | 0 \le t \le 1\}$. Take $x = t$, $y = t$, $z = 0$, $dx = dt$, then $\int_{C_1} xydx + yzdy + xzdz = \int_0^1 t^2 dt = \frac{1}{3}$
2. $C_2 = \{i + j + tk | 0 \le t \le 1\}$. Take $x = 1$, $y = 1$, $z = t$, $dx = dy = 0$, $dz = 1$, then $\int_{C_2} xydx + yzdy + xzdz = \int_0^1 tdt = \frac{1}{2}$
3. $\int_{C_1 \bigcup C_2} xydx + yzdy + zxdz = \int_{C_1} xydx + yzdy + zxdz + \int_{C_2} xydx + yzdy + zxdz = \frac{5}{6}$

(4) 試求向量場 $\vec{F}$ 的位勢函數 (potential function),

$$\vec{F} = \frac{y}{1+x^2y^2}\vec{i} + (\frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}})\vec{j} + (\frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z})\vec{k}$$

(10分)

(5) 試求向量場 $\vec{F} = (3xy - \frac{x}{1+y^2})\vec{i} + (e^x + \tan^{-1} y)\vec{j}$ 由心臟曲線 $r = 3(1+\cos\theta)$ 向外流出總量 (outward flux). (12分)

Sol. $M = 3xy - \frac{x}{1+y^2}$, $N = e^x + tan^{-1}y \Rightarrow \frac{\partial M}{\partial x} = 3y - \frac{1}{1+y^2}$, $\frac{\partial N}{\partial y} = \frac{1}{1+y^2} \Rightarrow$ Flux= $\int\int_R (3y - \frac{1}{1+y^2} + \frac{1}{1+y^2})dxdy = \int\int_R 3y\, dxdy = \int_0^{2\pi}\int_0^{a(1+\cos\theta)} (3r\sin\theta) rdrd\theta = \int_0^{2\pi} a^3(1+\cos\theta)^3(\sin\theta)d\theta = [-\frac{a^3}{4}(1+\cos\theta)^4]_0^{2\pi} = -4a^3 - (-4a^3) = 0$

(6) 求 $\int\int_S \vec{F}\cdot\vec{n}d\sigma$, 其中 $\vec{F} = (x^2+y^2)\vec{k}$, S 爲曲面 $(z+xy)^3 = x^2+y^2$ 限制在 $1 \le x^2+y^2 \le 4$, $\vec{n}$ 爲往上之法向量 (即 $\vec{n}$ 沿 $\vec{k}$ 之分量 ≥ 0). (12分)

Sol. S爲$f(x,y,z) = (z+xy)^3 - (x^2+y^2)$之等位面.

$$\nabla f = (3(z+xy)^2y - 2x, 3(z+xy)^2x - 2y, 3(z+xy)^2),$$

$3(z+xy)^2 = 3(x^2+y^2)^{\frac{2}{3}} \ge 0$. $\vec{n}d\sigma = \frac{\nabla f}{|\nabla f\cdot\vec{k}|}dxdy = \frac{\nabla f}{3(x^2+y^2)^{\frac{2}{3}}}dxdy$.
$\vec{F}\cdot\vec{n}d\sigma = (x^2+y^2)\cdot 3(x^2+y^2)^{\frac{2}{3}}\cdot\frac{dxdy}{3(x^2+y^2)^{\frac{2}{3}}} = (x^2+y^2)dxdy = r^3 drd\theta$
$\int\int_S \vec{F}\cdot\vec{n}d\sigma = \int_0^{2\pi} d\theta \int_1^2 r^3 dr = 2\pi\cdot\frac{(2^4-1)}{4}$

(7) 令曲面 S 表示球面 $x^2+y^2+z^2 = 4$ 被錐面 $z = \sqrt{x^2+y^2}$ 所截的部分. 試求面積分 $\int\int_S y^2 z d\sigma$. (12分)

Sol. 令$R: x^2+y^2 \le 2$, $z = \sqrt{4-x^2-y^2} = f(x,y)$, $d\sigma = \sqrt{1+(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2}dxdy = \frac{2}{\sqrt{4-x^2-y^2}}dxdy$, 令$x = r\cos\theta$, $y = r\sin\theta$, $0 \le r \le \sqrt{2}$, $0 \le \theta \le 2\pi$. 所以

$$\begin{aligned} \int\int_S y^2 z d\sigma &= \int\int_S y^2\sqrt{4-x^2-y^2}\cdot\frac{2}{\sqrt{4-x^2-y^2}}dxdy \\ &= 2\int\int_S y^2 dxdy = 2\int_0^{\sqrt{2}}\int_0^{2\pi} r^2\cos^2\theta rd\theta dr \\ &= 2\int_0^{\sqrt{2}} r^3 dr \int_0^{2\pi}\cos^2\theta d\theta = 2\pi \end{aligned}$$

(8) 設 S 是柱面 $x^2+y^2=1,\ 0\le z\le 1$, 再加上頂蓋 $x^2+y^2\le 1, z=1$ 所形成; 且設 $\vec{F}=-y\vec{i}+x\vec{j}+x^2\vec{k}$. 試求 $\int\int_S \nabla\times\vec{F}\cdot\vec{n}\,d\sigma$ 之值. (10分)

Sol.

解一 用Stoke's 定理, C爲$x^2+y^2=1,\ z=0,\ r=\cos ti+\sin tj,\ r'=-\sin ti+\cos tj \Rightarrow \int\int_S \nabla\times F\cdot n d\sigma=\oint_C F\cdot dr=\int_0^{2\pi}1dt=2\pi$.

解二 若直接計算比較麻煩, 此時令柱面部分S_1, 外法向量n_1, 頂面部分S_2, 外法n_2. 可得$\int\int_{S_1}\nabla\times F\cdot n_1 d\sigma=0$ (因爲$\nabla\times F\cdot n_1$=-2xy). $\nabla\times F\cdot n_2=2$. $\int\int_{S_2}\nabla\times F\cdot n_2 d\sigma=\int\int_{S_2}2d\sigma=2\pi$

(9) 設曲面S爲$x^2+y^2+z^2=1$, $\vec{n}$爲向外單位法向量. $\vec{F}(x,y,z)=3xy^2\vec{i}+3x^2y\vec{j}+z^3\vec{k}$. 求曲面積分$\int\int_S \vec{F}\cdot\vec{n}d\sigma$. (12分)

Sol. 利用 Gauss 定理: 令B爲$x^2+y^2+z^2\le 1$

$$\int\int_S \vec{F}\cdot\vec{n}d\sigma=\int\int\int_B(\nabla\cdot\vec{F})dv$$

$$\begin{aligned}\nabla\cdot\vec{F} &= \frac{\partial}{\partial x}(3x^2y)+\frac{\partial}{\partial y}3xy^2+\frac{\partial}{\partial z}(z^3)\\ &= 3y^2+3x^2+3z^2=3(x^2+y^2+z^2)\end{aligned}$$

$$\int\int\int_B(\nabla\cdot\vec{F})dv=3\int\int\int_B(x^2+y^2+z^2)dxdydz$$

引入球座標$\begin{cases} x = \rho\sin\phi\cos\theta\\ y = \rho\sin\phi\sin\theta\\ z = \rho\cos\phi\end{cases}$

$$\begin{aligned} &= 3\int_0^1\int_0^\pi\int_0^{2\pi}\rho^2\cdot\rho^2\sin\phi d\theta d\phi d\rho\\ &= 3\int_0^1\rho^4 d\rho\int_0^{2\pi}\int_0^\pi\sin\phi d\phi d\theta=\frac{12}{5}\pi\end{aligned}$$