微甲 II 組 (07-12班)-考題部分

1. 設$f(x) = \frac{lnx}{x}, x > 0.$

(i) 試求 f 的最高點. (5分)

(ii) 試證 $\pi^e < e^\pi$. (5分)

解

(i) $f(x) = \frac{\ln x}{x}$, $x > 0$, $f'(x) = \frac{1-\ln x}{x^2}$.
On $x > e$ $\ln x > 1$, $f'(x) < 0$, $f(x)$ decreasing on $(0, \infty)$.
On $0 < x < e$, $\ln x < 1$, $f'(x) > 0$, $f(x)$ increasing on $(0, e)$.
Hence $f(e) = \frac{\ln e}{e} = \frac{1}{e}$ is the absolutly maximum value of f.

(ii) $f(e) > f(\pi)$, $\frac{\ln e}{e} > \frac{\ln \pi}{\pi}$, $\pi \ln e > e \ln \pi$, $e^\pi > \pi^e$.

2. 設 $f(x) = e^{g(x)}$ 及 $g(x) = \int_2^x \frac{t}{1+t^4}\, dt$. 試求 $f'(2)$. (10分)

解 $f'(x) = \frac{x}{1+x^4}$. $f'(2) = \frac{2}{17}$.

3. Let

$$f(x) = \ln x - \frac{1}{8}x^2, 1 \le x \le 2$$

Find the length L of the graph of f. (10分)

解 Since $f'(x) = \frac{1}{x} - \frac{x}{4}$, $L = \int_1^2 \sqrt{1 + (\frac{1}{x} - \frac{x}{4})^2} dx = \int_1^2 \sqrt{1 + (\frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{16})} dx =$
$\int_1^2 \sqrt{\frac{1}{x^2} + \frac{1}{2} + \frac{x^2}{16}} dx = \int_1^2 \sqrt{(\frac{1}{x} + \frac{x}{4})^2} dx = \int_1^2 (\frac{1}{x} + \frac{x}{4}) dx = (\ln x + \frac{x^2}{8})|_1^2 =$
$\ln 2 + \frac{3}{8}$

4. 求積分

(i) $\int \frac{x^2+3x+2}{x^3-3x+2}\, dx$ (8分)

(ii) $\int_1^e \frac{1}{x\sqrt{1+(\ln x)^2}}\, dx$ (7分)

5. 研判瑕積分 $\int_0^2 \frac{dx}{(1-x)^2}$ 是收斂或發散 . (10分)

解 $\int_0^2 \frac{dx}{(1-x)^2} = \int_{-1}^0 \frac{1}{t^2} dt$. (t=1-x)

$=\lim_{x\to 0} \frac{-1}{t}|_{-1}^x$, it's divergent .

6. 若無窮級數 $\sum_{k=3}^{\infty} \frac{1}{k(lnk)[\ln(lnk)]^p}$收斂, 則 p 的最大範圍爲何?(10分)

解 使用積分檢定法, 考慮$a_k = f(k)$, 此處$f(x) = \frac{1}{x(\ln x)[\ln(\ln x)]^p}$. 令$u = \ln x, \int f(x)dx = \int \frac{du}{u(\ln u)^p} = \int \frac{dv}{v^p}$(令$v = \ln u$). 故當$p > 1$時,$\int \frac{dv}{v^p}$收斂 ; 而當$p \leq 1$時,$\int \frac{dv}{v^p}$發散 .

7. (i) 試說明$\sum_{n=1}^{\infty} \frac{n!}{n^n}$爲收斂級數.(5分)

(ii) 設$a > 0$爲一正數. 試求$f(t) = e^{at} - 1$之 Maclaurin 級數.(5分)

(iii) 試說明$\sum_{n=1}^{\infty}(\sqrt[n]{2} - 1)$爲發散級數 (可利用 (ii)).(5分)

解

(i) 用 ratio test $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = (\frac{n}{n+1})^n = (1 - \frac{1}{n+1})^n \to \frac{1}{e}$, as $n \to \infty$

(ii) $e^{at} - 1 = at + \frac{(at)^2}{2!} + ... + \frac{(at)^n}{n!} + ...$

(iii) $\sqrt[n]{2} - 1 = e^{\frac{1}{n}\ln 2} - 1 > (\ln 2) \cdot \frac{1}{n}, n = 1, 2, ...$而$\sum(\ln 2) \cdot \frac{1}{n}$爲發散, 所以$\sum(\sqrt[n]{2} - 1)$爲發散.

8. (i) 試寫出 $g(t) = \cos t$ 之 Maclaurin 級數.(5分)

(ii) 試將$\int_0^1 \cos x^2 dx$表爲一交錯級數之和並估計其值至誤差小於$\frac{1}{1000}$.(5分)

解

(i) $\cos t = 1 - \frac{t^2}{2} + \frac{t^4}{4!} + ... + (-1)^n \frac{t^{2n}}{(2n)!} + ...$

(ii) $\int_0^1 \cos^2 x dx = \int_0^1 1 - \frac{x^4}{2} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + ...dx = 1 - \frac{1}{5 \cdot 2!} + \frac{1}{9 \cdot 4!} - \frac{1}{13 \cdot 6!} ...$, 而$\frac{1}{13 \cdot 6!} < \frac{1}{1000}$, 所以$\int_0^1 \cos x^2 dx \approx 1 - \frac{1}{10} + \frac{1}{216}$

9. 求下列級數的收斂區間 (注意要討論在兩個端點上, 級數是否收斂)

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n \text{ . (10分)}$$

解 By the ratio test, $\lim_{x \to \infty} |\frac{(n)(x)^{n+1}}{|(n+1)(x)^n|}| = |x|$, so, if $|x| < 1$ then it's abs. conv ; if $|x| > 1$, then it is div. so the radius of conv is $r = 1$
If $x = 1$,then the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges ,by the integral test .
If $x = -1$,then the alternating series $\sum_{n=1}^{\infty} \frac{-1^n}{n}$ converges ,since $\lim_{n \to \infty} \frac{1}{n}$ = 1.