1. (10%) Evaluate the iterated integral $\int_0^a \int_x^a \sin(y^2)\, dy\, dx,\ a > 0.$

Sol:

$$\begin{aligned}\int_0^a \int_x^a \sin y^2\, dy\, dx &= \int_0^a \int_0^y \sin y^2\, dx\, dy \\ &= \int_0^a y \sin y^2\, dy \\ &= \frac{-\cos y^2}{2}\Big|_{y=0}^{y=a} \\ &= \frac{1-\cos a^2}{2}\end{aligned}$$

2. (12%) Compute the area of the domain in the first quadrant bounded by the four curves $xy = 1,\ xy = 4,\ \frac{y}{x^2} = 1,$ and $\frac{y}{x^2} = 2.$

Sol:

Method 1

(i) Find the four vertices of the region:

$(2^{-\frac{1}{3}}, 2^{\frac{1}{3}}), (1,1), (2^{\frac{1}{3}}, 2^{\frac{5}{3}}), (2^{\frac{2}{3}}, 2^{\frac{4}{3}}).$

(From left to right.)

(ii)

$$\begin{aligned}\text{Area } &= \int_{2^{-\frac{1}{3}}}^{1} (2x^2 - \frac{1}{x})\, dx \\ &+ \int_1^{2^{\frac{1}{3}}} (2x^2 - x^2)\, dx \\ &+ \int_{2^{\frac{1}{3}}}^{2^{\frac{2}{3}}} (\frac{4}{x} - x^2)\, dx \\ &= \ln 2\end{aligned}$$

or

$$\begin{aligned}
\text{Area} &= \int_1^{2^{\frac{1}{3}}} (y^{\frac{1}{2}} - y^{-1})\, dy \\
&+ \int_{2^{\frac{1}{3}}}^{2^{\frac{4}{3}}} (y^{\frac{1}{2}} - (\frac{y}{2})^{\frac{1}{2}})\, dy \\
&+ \int_{2^{\frac{4}{3}}}^{2^{\frac{5}{3}}} (\frac{4}{y} - (\frac{y}{2})^{\frac{1}{2}})\, dy \\
&= \ln 2.
\end{aligned}$$

Method 2

(i) Make the change of variables:

$$\begin{cases} u &= xy \\ v &= \frac{y}{x}2 \end{cases}$$

$$\Rightarrow \begin{cases} x &= (\frac{u}{v})^{\frac{1}{3}} \\ y &= u^{\frac{2}{3}} v^{\frac{1}{3}} \end{cases}$$

(ii)

$$\text{Area} = \int_1^2 \int_1^4 |\frac{\partial(x,y)}{\partial(u,v)}| dudv = \ln 2.$$

3. (12%) Find the region $E \subset \mathbb{R}^3$ for which the triple integral $\iiint_E (4 - x^2 - 4y^2 - 9z^2)\, dV$ is a maximum, and compute this maximum value.

Sol:

First we note that the region E which maximize the integral $\int 4 - x^2 - 4y^2 - 9z^2$ is the ellipsoid

$$(x, y, z) : 4 - x^2 - 4y^2 - 9z^2 > 0$$

Here we use the "change of coordinate", that is

$$x = r \sin\phi \cos\theta, y = \frac{1}{2} r \sin\phi \sin\theta, z = \frac{1}{3} r \cos\phi$$

and the corresponding Jacobian is

$$\frac{\partial(x, y, z)}{\partial(r, \phi, \theta)} = \frac{1}{6} r^2 \sin\phi$$

With the above transformation, the original integral $\int_{x^2+4y^2+9z^2<4} 4 - x^2 - 4y^2 - 9z^2 \, dxdydz$ is now changed to

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{2} (4 - r^2)\frac{1}{6} r^2 \sin\phi \, drd\phi d\theta = \frac{128}{45}\pi$$

4. (14%) Let $\mathbf{F}(x, y, z) = yz\mathbf{i} + \left(xz + \frac{y}{y^2 + z^2 + 1}\right)\mathbf{j} + \left(xy + \frac{z}{y^2 + z^2 + 1} + \cos z\right)\mathbf{k}$.

   (a) Find a function f such that $\mathbf{F} = \nabla f$.

   (b) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve starting from the origin given by $\mathbf{r}(\theta) = \langle \theta\cos\theta, \theta\sin\theta, \theta \rangle$, $0 \le \theta \le \pi$.

Sol:

(a) $\frac{\partial f}{\partial x} = yz \Rightarrow f = xyz + g(y, z)$

$\frac{\partial f}{\partial x} = xz + \frac{y}{y^2 + z^2 + 1} \Rightarrow f = xyz + \frac{1}{2}\ln(y^2 + z^2 + 1) + h(z)$

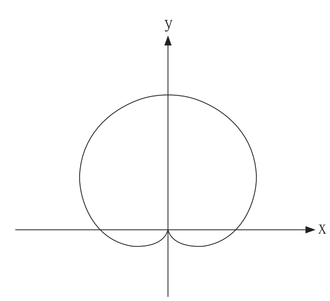
$\frac{\partial f}{\partial x} = xy + \frac{z}{y^2 + z^2 + 1} + \cos y \Rightarrow f = xyz + \frac{1}{2}\ln(y^2 + z^2 + 1) + \sin z$

(b)

$$\int_C F \cdot dr = f(x, y, z)\Big|_{r(0)=(0,0,0)}^{r(\pi)=(-\pi,0,\pi)} = \frac{1}{2}\ln(\pi^2 + 1)$$

5. (14%) Compute the line integral $\oint_C (3xy + 1)\, dx + (x^2 + x)\, dy$, where the closed curve C is the cardioid given by the polar equation $r = 1 + \sin\theta$, and is oriented counterclockwise (see figure).

Sol:



Since $3xy + 1$, $x^2 + x$ are smooth on C and smooth in D ,where D is the region enclosed by C. Hence using Green's theorem we have

$$\oint_c (3xy + 1)dx + (x^2 + x)dy = \iint_D \frac{\partial(x^2 + x)}{\partial x} - \frac{\partial(3xy + 1)}{\partial y} dA = \iint_D (1 - x)dA$$

by using polar coordinates we have

$$\iint_D (1-x)dA = \int_0^{2\pi}\int_0^{1+\sin\theta}(1-r\cos\theta)rdrd\theta$$

$$\begin{aligned}\int_0^{2\pi}\int_0^{1+\sin\theta}(1-r\cos\theta)rdrd\theta &= \int_0^{2\pi}[\frac{(1+\sin\theta)^2}{2}-\frac{1}{3}\cos\theta(1+\sin\theta)^3]d\theta\\ &= \int_0^{2\pi}(\frac{1}{2}+\sin\theta+\frac{1}{2}\sin^2\theta)d\theta\\ &= \pi+\frac{\pi}{2}=\frac{3\pi}{2}\end{aligned}$$

6. (10%) Evaluate $\iint\limits_S \sqrt{1+x^2+y^2}\,dS$, where S is the helicoid parametrized as $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k},\ 0\le u\le 1,\ 0\le v\le\pi$.

Sol:

$r(u,v) = (u\cos v, u\sin v, v)$

$\frac{\partial r}{\partial u} = (\cos v, \sin v, v)$

$\frac{\partial r}{\partial v} = (-u\sin v, u\cos v, 1)$

$\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v} = (\sin v, -\cos v, u)$

$\left|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right| = \sqrt{1+u^2}$

$\iint\limits_S \sqrt{1+x^2+y^2}dS = \int_0^1\int_0^\pi \sqrt{1+u^2}\sqrt{1+u^2}dvdu = (u+\frac{1}{3}u^3)\Big|_0^1 (v)\Big|_0^\pi = \frac{4}{3}\pi$

7. (14%) Let closed curve C be the intersection of the plane $x+y+z=1$ and the cylinder $x^2+y^2=9$, oriented counterclockwise as viewed from above, and $\mathbf{F}(x,y,z) = x^2z\mathbf{i}+xy^2\mathbf{j}+z^2\mathbf{k}$. Find $\oint_C \mathbf{F}\cdot d\mathbf{r}$.

Sol:

Compute the curl of vector field $\mathbf{F}$. Find $\mathbf{curl}(\mathbf{F}) = x^2\mathbf{j}+y^2\mathbf{k}$. Using Stoke's Theorem, we have

$$\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \iint_{\mathcal{S}} \mathbf{curl}(\mathbf{F})d\mathbf{S}$$

where $\mathcal{S}$ is the surface with boundary $\mathcal{C}$ which is parametrized with counterclockwise orientation. And $\mathcal{C}$ is the intersection of $x^2+y^2=9$ and $x+y+z=1$. Now observe that the normal vector of $\mathcal{S}$ is $(1,1,1)/\sqrt{3}$ and that $d\mathcal{S} = \sqrt{3}\, d\mathcal{A}$. So

$$\iint_{\mathcal{S}} \mathbf{curl}(\mathbf{F}) = \iint_{\mathcal{A}} (x^2+y^2) d\mathcal{A}$$

where $\mathcal{A}$ is the projection of $\mathcal{S}$ on xy plane., that is, $x^2+y^2 = 9$. Hence by using polar coordinates. We have

$$\iint_{\mathcal{A}} (x^2+y^2) d\mathcal{A} = \int_0^3 \int_0^{2\pi} r^2 r dr d\theta = \frac{81}{2}\pi$$

8. (14%) Let $\mathbf{F}(x,y,z) = \left(xy^2 + \sqrt{y^2+z^4}\right)\mathbf{i} + \left(\tan^{-1} x + x^2 y\right)\mathbf{j} + \left(\frac{z^3}{3} - e^{x^2+y^2}\right)\mathbf{k}$.

   (a) Find div $\mathbf{F}$.

   (b) Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where the surface S is the top half of the sphere $x^2+y^2+z^2 = 1$ with the unit normal vectors pointing away from the origin. *Warning.* S is *not* a closed surface!

Sol:

(a) div$\mathbf{F} = x^2+y^2+z^2$

(b) 1° By divergence theorem, we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} + \iint_E \mathbf{F} \cdot d\mathbf{S} = \iiint x^2+y^2+z^2 dxdydz$$

where E is the disk with radius 1 on xy-plane and with the unit normal vectors $(0,0,-1)$.

2°

$$\iint_E \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2+y^2\leq 1} e^{x^2+y^2} - \overset{0}{\cancel{\frac{z^3}{3}}} dxdy = \int_0^{2\pi} \int_0^1 e^{r^2} \cdot r dr d\theta = \pi(e-1)$$

3°

$$\iiint x^2+y^2+z^2 dxdydz = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \cdot \rho^2 \sin\phi d\rho d\phi d\theta = \frac{2\pi}{5}$$

4° Thus

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{2\pi}{5} - \pi(e-1)$$