972 微甲01-05班期中考解答

1. (8%) Determine whether the series converges absolutely, or converges conditionally, or diverges.

   (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{3\cdot 5\cdot 7\cdots(2n+1)}{n!\cdot 3^n}$.

   (b) $\displaystyle\sum_{n=1}^{\infty}\ln(1+\frac{1}{\sqrt{n}})$.

   Sol:

   (a) Let $a_n = (-1)^n \dfrac{3\cdot 5\cdot 7\cdots(2n+1)}{n!\cdot 3^n}$, applying *Ratio Test*,

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{3\cdot 5\cdot 7\cdots(2n+1)\cdot(2n+3)}{(n+1)!\cdot 3^{n+1}}\cdot\frac{n!\cdot 3^n}{3\cdot 5\cdot 7\cdots(2n+1)}\right| \\
&= \lim_{n\to\infty}\left|\frac{2n+3}{3\cdot(n+1)}\right| \\
&= \frac{2}{3} < 1
\end{aligned}$$

   So, $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{3\cdot 5\cdot 7\cdots(2n+1)}{n!\cdot 3^n}$ is **absolutely convergent**.

   (b) Let $a_n = \ln(1+\dfrac{1}{\sqrt{n}})$, $b_n = \dfrac{1}{\sqrt{n}}$,

   $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\ln(1+\frac{1}{\sqrt{n}})}{\frac{1}{\sqrt{n}}} = 1$ (L'Hospital's rule)

   Since $\sum b_n$ is a divergent series (p-series with $p=\frac{1}{2}$), by *Limit Comparison Test*, $\sum a_n$ is also **divergent**.

   Note: The Limit Comparison Test must compare two positive series. We cannot compare $\sum a_n$ with $\sum \ln(\frac{1}{\sqrt{n}})$

2. (4%) Determine the real values of p for which the series $\displaystyle\sum_{n=3}^{\infty}(-1)^n\frac{1}{n^p\ln n(\ln\ln n)^2}$ is absolutely convergent, conditionally convergent or divergent, respectively.

   Sol:

   Consider the series $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n^p\ln n(\ln\ln n)^2}$

$\underline{p<0}$: $\displaystyle\lim_{n\to\infty}\frac{1}{n^p\ln n(\ln\ln n)^2}=\lim_{n\to\infty}\frac{n^{-p}}{\ln n(\ln\ln n)^2}\to\infty\neq 0.$

So the series is **divergent**.

$\underline{p=1}$: Using *Integral Test*,

$$\int_3^\infty \frac{1}{x\ln x(\ln\ln x)^2}\,dx \overset{u=\ln\ln x}{=} \int_{\ln\ln 3}^\infty \frac{1}{u^2}\,du = -\frac{1}{u}\bigg|_{\ln\ln 3}^\infty < \infty$$

So, the series is **absolutely convergent**.

$\underline{p>1}$: Using *Comparison Test*, comparing with p-series ($p>1$)

$\displaystyle\sum_{n=3}^\infty \frac{1}{n^p\ln n(\ln\ln n)^2} < \sum_{n=3}^\infty \frac{1}{n^p} < \infty.$

So the series is **absolutely convergent**.

$\underline{0\le p<1}$: Compare with $\displaystyle\sum\frac{1}{n}$, $\displaystyle\lim_{n\to\infty}\frac{\frac{1}{n^p\ln n(\ln\ln n)^2}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n^{1-p}}{\ln n(\ln\ln n)^2}\to\infty,$

Since $\displaystyle\sum\frac{1}{n}$ is divergent, $\displaystyle\sum\frac{1}{n^p\ln n(\ln\ln n)^2}$ is divergent.

However, $\displaystyle\sum(-1)^n\frac{1}{n^p\ln n(\ln\ln n)^2}$ is convergent by *Alternating Series Test*. Hence, the series is **conditionally convergent**.

3. (11%)

(a) Write down the Maclaurin series of $\arctan x$.

(b) What is the interval of convergence of the above series?

(c) Find the sum of the series $\displaystyle\sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)3^n}$.

Sol:

(a) Since $\displaystyle\frac{d}{dx}\arctan x=\frac{1}{1+x^2}$, $\displaystyle\arctan x = C+\sum_{n=0}^\infty(-1)^n\frac{x^{2n+1}}{2n+1}$. Because $\arctan 0=0$, we have $C=0$, therefore $\displaystyle\arctan x=\sum_{n=0}^\infty(-1)^n\frac{x^{2n+1}}{2n+1}$.

(b) Since $\lim_{n\to\infty} \sqrt[n]{|\frac{x^{2n+1}}{2n+1}|} = x^2$, this series converges absolutely when $-1 < x < 1$. For $x = \pm 1$, this series also converge (conditionally) by Leibniz theorem. Hence the interval of convergence of this seris is $[-1, 1]$.

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = \sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} (\frac{1}{\sqrt{3}})^{2n+1} = \sqrt{3} \arctan \frac{1}{\sqrt{3}} = \frac{\sqrt{3}\pi}{6}$.

4. (11%) Let $\mathbf{r}(t)$ be a motion governed by Newton's Law $\mathbf{F} = m\mathbf{a}$ and the Gravitational Law $\mathbf{F} = -\frac{GMm}{r^3}\mathbf{r}$ where $r = |\mathbf{r}|$.

(a) Show that $\mathbf{r} \times \mathbf{v}$ is a constant vector $\mathbf{h}$. Deduce that the orbit of the particle is a plane curve.

(b) Let $\mathbf{u} = \frac{\mathbf{r}}{r}$. Show that $\mathbf{a} \times \mathbf{h} = GM\mathbf{u}'$. Deduce that there is a constant vector $\mathbf{c}$ such that $\mathbf{v} \times \mathbf{h} = GM\mathbf{u} + \mathbf{c}$. (Hint: use the formula $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.)

Sol:

(a) By $\mathbf{F} = m\mathbf{a} = -\frac{GMm}{r^3}\mathbf{r}$, we obthain

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

So, $\mathbf{a}$ is parallel to $\mathbf{r}$. Thus

$$\begin{aligned} \frac{d}{dt}(\mathbf{r} \times \mathbf{v}) &= \mathbf{r}' \times \mathbf{v} + \mathbf{r} \times \mathbf{v}' \\ &= \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} = 0 + 0 = 0 \end{aligned}$$

Therefore $\mathbf{r} \times \mathbf{v} = \mathbf{h}$ is a constant vector.

(We may assume that $\mathbf{h} \neq 0$; that is, $\mathbf{r}$ and $\mathbf{v}$ are not parallel.)
This means that the vector $\mathbf{r} = \mathbf{r}(t)$ is perpendicular to $\mathbf{h}$ for all values of t, so the planet always lies in the plane through the origin perpendicular to $\mathbf{h}$. Thus the orbit of the planet is a plane curve.

(b)

$$\begin{aligned}
\mathbf{h} &= \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \mathbf{r}' = r\mathbf{u} \times (r\mathbf{u})' \\
&= r\mathbf{u} \times (r\mathbf{u}' + r'\mathbf{u}) = r^2 \mathbf{u} \times \mathbf{u}' + rr'(\mathbf{u} \times \mathbf{u}) \\
&= r^2 \mathbf{u} \times \mathbf{u}'
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{a} \times \mathbf{h} &= -\frac{GM}{r^2}\mathbf{u} \times (r^2 \mathbf{u} \times \mathbf{u}') = -GM\mathbf{u} \times (\mathbf{u} \times \mathbf{u}') \\
&= -GM[(\mathbf{u} \cdot \mathbf{u}')\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{u}']
\end{aligned}$$

But $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 = 1$, it follows from Example 4 in section 13.2 that $\mathbf{u} \cdot \mathbf{u}' = 0$. Therefore

$$\mathbf{a} \times \mathbf{h} = GM\mathbf{u}'$$

and since $\mathbf{h}$ is a constant vector,

$$(\mathbf{v} \times \mathbf{h})' = \mathbf{v}' \times \mathbf{h} = \mathbf{a} \times \mathbf{h} = GM\mathbf{u}'$$

Integrating both side of this equation, we get

$$\mathbf{v} \times \mathbf{h} = GM\mathbf{u} + \mathbf{c}$$

,for some constant vector $\mathbf{c}$.

5. (11%) Let $\mathbf{r}(t) = 2\sin t\mathbf{i} + 5t\mathbf{j} + 2\cos t\mathbf{k},\ t \in \mathbb{R}$.

   (a) Find the unit tangent vector $\mathbf{T}(t)$, unit normal vector $\mathbf{N}(t)$ and binormal vector $\mathbf{B}(t)$.

   (b) Find the curvature κ.

Sol:

(a) Since $\mathbf{r}'(t) = 2cost\mathbf{i} + 5\mathbf{j} - 2\sin t\mathbf{k}$. $|\mathbf{r}'(t)| = \sqrt{4(\cos t)^2 + 4(\sin t)^2 + 25} = \sqrt{29}$
We have $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{29}}(2\cos t\mathbf{i} + 5\mathbf{j} - 2\sin t\mathbf{k})$.
$\mathbf{T}'(t) = \dfrac{1}{\sqrt{29}}(-2\sin t\mathbf{i} - 2\cos t\mathbf{k})$. $|\mathbf{T}'(t)| = \sqrt{\dfrac{4}{29}((\cos t)^2 + (\sin t)^2)} = \sqrt{\dfrac{4}{29}}$.

We have $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = -\sin t\mathbf{i} - \cos t\mathbf{k}$.

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2\cos t}{\sqrt{29}} & \frac{5}{\sqrt{29}} & \frac{-2\sin t}{\sqrt{29}} \\ -\sin t & 0 & -\cos t \end{vmatrix} = \frac{1}{\sqrt{29}}(-5\cos t\mathbf{i} + 2\mathbf{j} + 5\sin t\mathbf{k}).$$

(b) By the definition we have $\kappa(t) = \left|\dfrac{d\mathbf{T}}{dS}\right| = \left|\dfrac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right| = \dfrac{\sqrt{\frac{4}{29}}}{\sqrt{29}} = \dfrac{2}{29}$

6. (11%) Determine whether the function is continuous at $(0,0)$.

(a) $f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$

(b) $f(x,y) = \begin{cases} \dfrac{x^2y}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$

Sol:

(a) Since $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \lim_{r\to 0} \frac{r^3(\sin\theta(\cos\theta)^2)}{r^2}$. let $x = r\cos\theta$, $y = r\sin\theta$

$= \displaystyle\lim_{r\to 0} r(\sin\theta(\cos\theta)^2) = 0 = f(0,0)$. ($\because$ $|\sin\theta(\cos\theta)^2|$ is bounded by 1.)

By the definition we know that the function is continuous at $(0,0)$.

(b) Along the path $y = x$.

We have $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{(x^2+y^2)^2} = \lim_{(x,y)\to(0,0)} \frac{x^3}{4x^4} = \lim_{(x,y)\to(0,0)} \frac{1}{4x} \neq 0 = f(0,0)$.

The function isn't continuous at $(0,0)$.

7. (11%) Let $f(x,y) = \begin{cases} x^2\tan^{-1}\dfrac{y}{x} - y^2\tan^{-1}\dfrac{x}{y} & xy \neq 0 \\ 0 & x = 0 \text{ or } y = 0. \end{cases}$

(a) Find $f_x(0,0)$ and $f_x(0,y)$.

(b) Find $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$.

Sol:

(a)

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0.$$

For $y \neq 0$

$$\begin{aligned}
\frac{\partial f}{\partial x}(0,y) &= \lim_{h\to 0}\frac{f(h,y)-f(0,y)}{h} = \lim_{h\to 0}\frac{h^2\tan^{-1}\frac{y}{h} - y^2\tan^{-1}\frac{h}{y} - 0}{h} \\
&= \lim_{h\to 0}(h\tan^{-1}\frac{y}{h} - \frac{y^2\tan^{-1}\frac{h}{y}}{h}) = -y^2\lim_{h\to 0}\frac{\tan^{-1}\frac{h}{y}}{h} \cdots\cdots \frac{0}{0} \\
&= -\,y^2\lim_{h\to 0}\frac{\frac{\frac{1}{y}}{(\frac{h}{y})^2+1}}{1} = -y\frac{1}{(\frac{0}{y})^2+1} = -y.
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\partial^2 f}{\partial y\partial x}(0,0) &= \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})|_{(0,0)} \\
&= \lim_{h\to 0}\frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h} \\
&= \lim_{h\to 0}\frac{-h-0}{h} = -1.
\end{aligned}$$

8. (11%) Find the parametric equation for the tangent line to the curve of intersection of $z = x^2+y^2$ and $4x^2+y^2+z^2 = 9$ at the point $(-1,1,2)$.

Sol:

Let $f(x,y,z) = x^2+y^2-z$, $g(x,y,z) = 4x^2+y^2+z^2$

This problem is to find the tangent line to the curve of intersection of

$$f(x,y,z) = 0 \text{ and } g(x,y,z) = 9$$

At (-1,1,2), the normal vectors of the tangent planes to the surface functions are

$$\nabla f = (2x,2y,-1) = (-2,2,-1), \nabla g = (8x,2y,2z) = (-8,2,4)$$

So the direction of the tangent line is

$$(-2,2,-1)\times(-8,2,4) = (10,16,12)$$

The parametric equation for the tangent line

$$x = -1 + 10t$$

$$y = 1 + 16t$$

$$z = 2 + 12t$$

9. (11%) Let $f(x, y) = x^4 + y^4 - 4xy + 1$.

   (a) Find and classify all the critical points of $f(x, y)$.

   (b) Find the absolute maximal and minimal values of $f(x, y)$ on the disk $x^2 + y^2 \le 1$.

Sol:

(a) It's easy to see that $\nabla f(x, y) = (4x^3 - 4y, 4y^3 - 4x)$. Solve $\nabla f(x, y) = 0$. We get $x^3 = y$, and $y^3 = x$. Thus $y^9 = y$, $y = 1, -1$, or 0. Using $x = y^3$, we have $x = 1, -1$, or 0. So the critical points are $(1, 1)$, $(-1, -1)$, and $(0, 0)$. To classify the three points, consider the Hessian matrix

$$H_f(a, b) = \begin{bmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{xy}(a, b) & f_{yy}(a, b) \end{bmatrix} = \begin{bmatrix} 12a^2 & -4 \\ -4 & 12b^2 \end{bmatrix}$$

Since $H_f(1, 1)$ is positive definite, $(1, 1)$ is a local minimum. By the symmetric property, $(-1, -1)$ is also a local minimum. Now $\det H_f(0, 0) < 0$, therefore $(0, 0)$ is a saddle point.

(b) Since there are no maximum and minimum inside $D : x^2 + y^2 \le 1$, it must occur on the boundary. Let $x = \cos\theta$, $y = \sin\theta$, where $\theta \in [0, 2\pi]$. $f(\cos\theta, \sin\theta) = \cos^4\theta + \sin^4\theta - 4\sin\theta\cos\theta + 1 = 2 - 2\sin 2\theta - \sin^2\theta/2$. By squaring f, we have

$$f(\cos\theta, \sin\theta) = -\frac{1}{2}(\sin 2\theta + 2)^2 + 4$$

, where $\theta \in [0, 4\pi]$. Thus f has maximum $7/2$ and minimum $-1/2$.

10. (11%) If the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $(a, b > 0)$ is to enclose the circle $x^2 + y^2 = 2y$, what values of a and b minimize the area of ellipse?

Sol:

$x^2 = 2y - y^2 \Rightarrow \dfrac{2y - y^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow (a^2 - b^2)y^2 + 2b^2 y - a^2 b^2 = 0$

y has only one solution $\Rightarrow 4b^2 + 4a^2(a^2 - b^2) = 0 \Rightarrow a^4 - a^2b^2 + b^2 = 0$.

Let $f(a,b) = \pi ab$ and $g(a,b) = a^4 - a^2b^2 + b^2 = 0$,

then $\nabla f = (\pi b, \pi a)$ and $\nabla g = (4a^3 - 2ab^2, -2a^2b + 2b)$.

Use Lagrange multiplier, we have

$$\begin{cases} \pi b = 4\lambda a^3 - 2\lambda ab^2 \\ \pi a = -2\lambda a^2 b + 2\lambda b \\ a^4 - a^2b^2 + b^2 = 0 \end{cases}$$

Solve this equation, we get $a = \dfrac{\sqrt{6}}{2}$, $b = \dfrac{3\sqrt{2}}{2}$, $\lambda = -\dfrac{\sqrt{3}}{3}\pi$, and the minimum of f is $f(\dfrac{\sqrt{6}}{2}, \dfrac{3\sqrt{2}}{2}) = \dfrac{3\sqrt{3}}{2}\pi$.