1. (11%) Evaluate the iterated integral $\int_0^1 \int_{\sqrt{x}}^1 \sin(y^3)\, dy\, dx$.

Sol:

$$\begin{aligned}
\int_0^1 \int_{\sqrt{x}}^1 \sin(y^3)\, dy\, dx &= \int_0^1 \int_0^{y^2} \sin(y^3)\, dx\, dy \\
&= \int_0^1 y^2 \sin y^3\, dy \\
&= -\frac{1}{3} \cos y^3\, dy \Big|_0^1 \\
&= \frac{1}{3} - \frac{1}{3}\cos 1
\end{aligned}$$

2. (11%) Evaluate $\oint_C (4y - e^{\cos x})\, dx + (13x + \sqrt{y^2+y+1})\, dy$, where $C$ is the circle $x^2+y^2=4$.

Sol:

$\oint_C (4y - e^{\cos x})\, dx + (13x + \sqrt{y^2+y+1})\, dy = \int_A (13-4)\, dx\, dy$, where $A$ is the region enclosed by $C$.

$= 9 \times 4\pi = 36\pi$.

3. (11%) Find the average value of the potential function $f(x,y,z) = \dfrac{mMG}{\sqrt{x^2+y^2+(z-2)^2}}$ on the spherical shell $r \leq \sqrt{x^2+y^2+z^2} \leq 1$, where $0 < r < 1$.

Sol:

$f_{average} = \dfrac{\iiint_E f(x,y,z)dV}{V}$ , where

$$\begin{aligned}
V &= \iiint_E 1dV \\
&= \int_0^{2\pi} \int_0^{\pi} \int_r^1 \rho^2 \sin\phi d\rho d\phi d\theta \\
&= \frac{4\pi}{3}(1 - r^3)
\end{aligned}$$

And the numerator is

$$\iiint\limits_E f(x,y,z)dV = \iiint\limits_E \frac{mMG}{\sqrt{x^2+y^2+(z-2)^2}}dV$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_r^1 \frac{mMG}{\sqrt{\rho^2-4\rho\cos\phi+4}}\rho^2\sin\phi d\rho d\phi d\theta$$
$$= 2\pi\cdot\int_0^{\pi}\int_r^1 \frac{1}{\sqrt{\rho^2-4\rho\cos\phi+4}}\rho^2\sin\phi d\rho d\phi$$
$$= (2\pi mMG)\int_r^1\int_0^{\pi} \frac{1}{\sqrt{\rho^2-4\rho\cos\phi+4}}\rho^2\sin\phi d\phi d\rho$$
$$= (2\pi mMG)\int_r^1 [\frac{\rho}{2}\sqrt{\rho^2-4\rho\cos\phi+4}]\ |_{\phi=0}^{\phi=\pi}\ d\rho$$
$$= (2\pi mMG)\int_r^1 \frac{\rho}{2}[(\rho+2)-|\rho-2|]d\rho$$
$$= (2\pi mMG)\int_r^1 \frac{\rho}{2}\cdot 2\rho d\rho$$
$$= (2\pi mMG)\int_r^1 \rho^2 d\rho$$
$$= (\frac{2}{3}\pi mMG)(1-r^3)$$
$$\therefore f_{average} = \frac{(\frac{2}{3}\pi mMG)(1-r^3)}{\frac{4\pi}{3}(1-r^3)}$$
$$= \frac{mMG}{2}$$

4. (11%) Evaluate the integral $\iint\limits_R \cos(x-y)\,dA$, where $R$ is the region bounded by $|x|+|y| = \frac{\pi}{2}$.

Sol:

$$I \equiv \iint_R \cos(x-y)dA \qquad R: |x|+|y| \leq \frac{\pi}{2}$$

method1:

Let $u = x+y$, $v = x-y$, then $|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}$

Therefore,

$$I \ = \ \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} \cos(v)\ \frac{1}{2}\ dvdu \ = \ \pi\ \sin(v)\Big|_{-\pi/2}^{\pi/2}\ \frac{1}{2} \ = \ \pi$$

method2:

$$\begin{aligned}
R_1 &= \int_0^{\pi/2}\int_0^{\pi/2-x} \cos(x-y)\,dydx = 1\\
R_2 &= \int_0^{\pi/2}\int_{-\pi/2+x}^{0} \cos(x-y)\,dydx = \frac{\pi}{2}-1\\
R_3 &= \int_{-\pi/2}^{0}\int_0^{\pi/2+x} \cos(x-y)\,dydx = \frac{\pi}{2}-1\\
R_4 &= \int_{-\pi/2}^{0}\int_{-\pi/2-x}^{0} \cos(x-y)\,dydx = 1\\
I &= R_1+R_2+R_3+R_4 = \pi
\end{aligned}$$

5. (11%) Evaluate $\iint_S y\,dS$, where $S$ is the surface $z = x + y^2,\ 0 \le x \le 1,\ 0 \le y \le 2$.

Sol:

Let $r = (x, y, x+y^2)$, then $r_x = (1,0,1), r_y = (0,1,2y)$

$r_x \times r_y = (-1,-2y,1)$ and $|r_x \times r_y| = \sqrt{2+4y^2}$

Thus, we have

$$\begin{aligned}
\iint_S y\,dS &= \int_0^1\int_0^2 y\sqrt{2+4y^2}\,dy\,dx\\
&= \int_0^1 \left[\frac{1}{8}\cdot\frac{2}{3}(2+4y^2)^{\frac{3}{2}}\right]_0^2 dx\\
&= \int_0^1 \frac{1}{12}\left[\sqrt{18^3}-\sqrt{2^3}\right] dx\\
&= \frac{13}{3}\sqrt{2}
\end{aligned}$$

6. (11%) Let $T$ be the surface of the tetrahedron bounded by $x=0,\ y=0,\ z=0$ and $\frac{x}{1}+\frac{y}{2}+\frac{z}{3}=1$, given with outward orientation. Let $\mathbf{F} = \frac{y^2}{z+2}\mathbf{i} + (xz+1)\mathbf{j} + (\frac{x^2}{y+2})\mathbf{k}$ be a vector field. Evaluate $\iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S}$, where $S$ is the surface when the bottom surface $(z=0)$ is removed from $T$.

Sol:

Method 1:

$$curl\mathbf{F} = (-x^2(y+2)^{-2} - x, -y^2(z+2)^{-2} - 2x(y+z)^{-1}, z - 2y(z+2)^{-1})$$

$$\begin{aligned}\iint_S curl\mathbf{F}\cdot d\mathbf{S} &= \iint_D curl\mathbf{F}(x,y,0)\cdot(0,0,1)dA = \iint_D -ydA \\ &= \int_0^1\int_0^{2-2x} -ydydx = \int_0^1 -\frac{1}{2}(2-2x)^2dx = -\frac{2}{3}\end{aligned}$$

Method 2:

By stoke's theorem,

$$\begin{aligned}\iint_S curl\mathbf{F}\cdot d\mathbf{S} &= \oint_C \mathbf{F}\cdot d\mathbf{r} \\ C = C_1\cup C_2\cup C_3 &= \{(t,0,0)\cup(1-t,2t,0)\cup(0,2-2t,0)\}, t\in[0,1] \\ \int_{C_1}\mathbf{F}\cdot d\mathbf{r} &= \int_0^1 0dt = 0 \\ \int_{C_2}\mathbf{F}\cdot d\mathbf{r} &= \int_0^1 -2t^2+2dt = \frac{4}{3} \\ \int_{C_3}\mathbf{F}\cdot d\mathbf{r} &= \int_0^1 -2dt = -2 \\ \Rightarrow \iint_S curl\mathbf{F}\cdot d\mathbf{S} &= 0+\frac{4}{3}-2 = -\frac{2}{3}\end{aligned}$$

7. (12%) (a) Let $\mathbf{F}_1(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|^3}$. Calculate $\iint_{S_1} \mathbf{F}_1\cdot d\mathbf{S}$, where $S_1$ is the sphere centered at $(0,0,0)$ with radius $a$. (You must show the details.)

   (b) Let $\mathbf{F}_2(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|^3} - \frac{\mathbf{x}-\mathbf{v}}{|\mathbf{x}-\mathbf{v}|^3}$ where $\mathbf{v} = \langle 2,0,0\rangle$. Find $\iint_{S_2} \mathbf{F}_2\cdot d\mathbf{S}$, where $S_2$ is the sphere centered at $(0,0,0)$ with radius $\frac{1}{2}$.

   (c) Find $\iint_{S_3} \mathbf{F}_2\cdot d\mathbf{S}$, where $S_3$ is the sphere centered at $(0,0,0)$ with radius 3.

Sol:

(a) $\mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|}$ for any $\mathbf{x}\in S_1$

$$\iint_{S_1} \mathbf{F}_1\cdot d\mathbf{S} = \iint_{S_1} \frac{\mathbf{x}}{|\mathbf{x}|^3}\cdot\frac{\mathbf{x}}{|\mathbf{x}|}dS = \iint_{S_1} \frac{1}{|\mathbf{x}|^2}dS = \frac{1}{a^2}\cdot Area(S_1) = 4\pi$$

(b) Let $\mathbf{F}_3(\mathbf{x}) = \mathbf{F}_1(\mathbf{x}) - \mathbf{F}_3(\mathbf{x}) = \dfrac{\mathbf{x}-\mathbf{v}}{|\mathbf{x}-\mathbf{v}|^3}$

$$
\begin{aligned}
div\mathbf{F}_1 &= div\langle \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \rangle \\
&= \frac{3\,(x^2+y^2+z^2)^{\frac{3}{2}} - x\cdot\frac{3}{2}\,(x^2+y^2+z^2)^{\frac{1}{2}}\cdot 2x}{(x^2+y^2+z^2)^3} \\
&+ \frac{-y\cdot\frac{3}{2}\,(x^2+y^2+z^2)^{\frac{1}{2}}\cdot 2y - z\cdot\frac{3}{2}\,(x^2+y^2+z^2)^{\frac{1}{2}}\cdot 2z}{(x^2+y^2+z^2)^3} \\
&= \frac{3\,(x^2+y^2+z^2) - 3x^2 - 3y^2 - 3z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} = 0 \quad \text{on } \mathbf{R}^3 \setminus \{(0,0,0)\}.
\end{aligned}
$$

Similiarly, $div\mathbf{F}_3 = 0$ on $\mathbf{R}^3 \setminus \{(2,0,0)\}$.

Therefore, we get $\displaystyle\iint_{S_2} \mathbf{F}_2 \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F}_1 \cdot d\mathbf{S} - \iiint_E div\mathbf{F}_3 dV = \iint_{S_2} \mathbf{F}_1 \cdot d\mathbf{S} = 4\pi$

by divergence theorem where $E$ is the ball bounded by $S$.

(c) Let $S^1$ be the sphere centered at (0,0,0) with radius $\frac{1}{2}$.

Let $S^2$ be the sphere centered at (2,0,0) with radius $\frac{1}{2}$.

Then by divergence theorem, $\displaystyle\iint_{S_3} \mathbf{F}_2 \cdot d\mathbf{S} = \iint_{S^1 \cup S^2} \mathbf{F}_2 \cdot d\mathbf{S}$.

$\displaystyle\iint_{S^1} \mathbf{F}_2 \cdot d\mathbf{S} = 4\pi$ as computed in case (b).

Similiarly, $\displaystyle\iint_{S^2} \mathbf{F}_2 \cdot d\mathbf{S} = -4\pi$.

Therefore, $\displaystyle\iint_{S_3} \mathbf{F}_2 \cdot d\mathbf{S} = 4\pi - 4\pi = 0$.

8. (11%) Find the general solution of the differential equation $y'' - 2y' + y = xe^x$.

Sol:

$y_{homo} = c_1e^x + c_2xe^x$ where $c_1, c_2$ are constants.

sub $y = Ax^4e^x$

we can obtain : $A = \dfrac{1}{6}$

hence , the general solution is: $y = c_1e^x + c_2xe^x + \dfrac{1}{6}x^4e^x$

9. (11%) Find all power series solutions to the (special case of) Bessel equation $x^2y''+xy'+x^2y = 0$.

Can $y(0)$ and $y'(0)$ be given arbitrarily?

Sol:

Let $y = \sum_{n=0}^{\infty} c_n x^n$, then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$.

$$\begin{aligned} x^2 y'' + x y' + x^2 y &= \sum_{n=2}^{\infty} n(n-1) c_n x^n + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} \\ &= \sum_{n=2}^{\infty} n(n-1) c_n x^n + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=2}^{\infty} c_{n-2} x^n \\ &= c_1 x + \sum_{n=2}^{\infty} (n(n-1) c_n + n c_n + c_{n-2}) x^n = 0. \end{aligned}$$

By comparing the coefficients, $c_1 = 0, n^2 c_n + c_{n-2} = 0, \forall n \geq 2$, i.e., $c_n = -\frac{c_{n-2}}{n^2}, \forall n \geq 2$.

Thus we have for all $k \in \mathbb{N}$, $\begin{cases} c_{2k} = (-\frac{1}{(2k)^2}) \cdot (-\frac{1}{(2k-2)^2}) \cdot \cdots \cdot (-\frac{1}{2^2}) c_0 = (-1)^k \frac{c_0}{2^{2k}(k!)^2} \\ c_{2k+1} = (-\frac{1}{(2k+1)^2}) \cdot (-\frac{1}{(2k-1)^2}) \cdot \cdots \cdot (-\frac{1}{3^2}) c_1 = 0 \end{cases}$.

Hence $y = \sum_{n=0}^{\infty} c_n x^n = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{4^k (k!)^2}$.

$y(0) = c_0$, can be given arbitrarily, $y'(0) = c_1 = 0$ is fixed, can't be given arbitrarily.