微甲08-13班 統一教學期末考解答

1. (10%) Evaluate the integral $\int x^\alpha \ln x\, dx$, $\alpha \in \mathbb{R}$.

   Sol:

   if $\alpha \neq -1$,

$$\int x^\alpha lnx dx = \int (\frac{x^{\alpha+1}}{\alpha+1})' lnx dx = (\frac{x^{\alpha+1}}{\alpha+1}) lnx - \int (\frac{x^{\alpha+1}}{\alpha+1})(lnx)' dx$$
$$= (\frac{x^{\alpha+1}}{\alpha+1}) lnx - \int \frac{x^\alpha}{\alpha+1} dx$$
$$= \frac{1}{\alpha+1} x^{\alpha+1} lnx - \frac{1}{(\alpha+1)^2} x^{\alpha+1} + c$$

   if $\alpha = -1$, let $t = lnx \Rightarrow \frac{dt}{dx} = \frac{1}{x}$,

$$\int x^{-1} lnx dx = \int t dt = \frac{1}{2} t^2 + c = \frac{1}{2}(lnx)^2 + c$$

2. (12%) Evaluate the integral $\int \frac{\sqrt{1+\sqrt{x}}}{x} dx$. (Hint. Make a substitution to express the integrand as a rational function.)

   Sol:

   $u = \sqrt{1+\sqrt{x}} \Rightarrow u^2 - 1 = \sqrt{x} \Rightarrow x = (u^2-1)^2 \Rightarrow dx = 2(u^2-1) \cdot 2u du$

$$\therefore \int \frac{\sqrt{1+\sqrt{x}}}{x} dx = \int \frac{u}{(u^2-1)^2} \cdot 2(u^2-1) \cdot 2u du = \int \frac{4u^2}{(u^2-1)} du$$
$$= \int \frac{4(u^2-1)}{(u^2-1)} du + 4 \int \frac{du}{u^2-1}$$
$$= \int 4 du + 2 \int \frac{1}{u-1} - \frac{1}{u+1} du$$
$$= 4u + 2\Big(\ln|u-1| - \ln|u+1| \Big) + C$$
$$= 4\sqrt{1+\sqrt{x}} + 2\Big(\ln(\sqrt{1+\sqrt{x}} - 1) - \ln(\sqrt{1+\sqrt{x}} + 1) \Big) + C$$

   or

$$= 4\sqrt{1+\sqrt{x}} + 2 \ln \frac{\sqrt{1+\sqrt{x}} - 1}{\sqrt{1+\sqrt{x}} + 1} + C.$$

3. (14%)

(a) Evaluate $\displaystyle\int \sec^3\theta\, d\theta$.

(b) Find the area of the surface obtained by rotating $y = \sin x$, $0 \le x \le \pi$, about the x-axis.

Sol:

(a)

$$\begin{aligned}
\because \int \sec^3 x dx &= \int \sec x d \tan x \\
&= \sec x \tan x - \int \tan x d \sec x \\
&= \sec x \tan x - \int \tan^2 x \sec x dx \\
&= \sec x \tan x - \int \left(\sec^2 x - 1\right) \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx
\end{aligned}$$

$\therefore \int \sec^3 x dx = \frac{1}{2}\left(\sec x \tan x + \int \sec x dx\right) = \frac{1}{2}\left(\sec x \tan x + \ln\left|\sec x + \tan x\right| + C\right)$.

(b) $y = \sin x$, $y\prime = \cos x$.

$$
\begin{aligned}
Area &= \int 2\pi y ds \\
&= \int_0^{\pi} 2\pi y \sqrt{1+(y')^2} dx \\
&= \int_0^{\pi} 2\pi \sin x \sqrt{1+\cos^2 x} dx \\
&= -2\pi \int_0^{\pi} \sqrt{1+\cos^2 x} d\cos x \\
&= -2\pi \int_0^{\pi} \sqrt{1+\cos^2 x} d\cos x \quad (Let \cos x = \tan\theta) \\
&= -2\pi \int_{\frac{\pi}{4}}^{\frac{-\pi}{4}} \sqrt{1+\tan^2\theta} d\tan\theta \\
&= 2\pi \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sec\theta d\tan\theta \\
&= 2\pi \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta d\theta \\
&= 2\pi \cdot \frac{1}{2} \left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right]_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \\
&= \pi \cdot \left[\sec\frac{\pi}{4}\tan\frac{\pi}{4} + \ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right| - \sec\frac{-\pi}{4}\tan\frac{-\pi}{4} - \ln\left|\sec\frac{-\pi}{4}+\tan\frac{-\pi}{4}\right|\right]_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \\
&= \pi \cdot \left[\sqrt{2}\cdot 1 + \ln\left|\sqrt{2}+1\right| - \sqrt{2}\cdot(-1) - \ln\left|\sqrt{2}-1\right|\right] \\
&= \pi \cdot \left[2\sqrt{2} + \ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|\right] \\
&= \pi \cdot \left[2\sqrt{2} + \ln\left(\sqrt{2}+1\right)^2\right] \\
&= 2\pi \cdot \left[\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right]
\end{aligned}
$$

4. (14%)

(a) Find the values of a for which the improper integral $\int_1^{\infty} \frac{dx}{x^a(1+\sqrt{x})}$ converges.

(b) Evaluate the integral $\int_0^{\pi} \sec^2 x\, dx$.

Sol:

(a) $\frac{1}{x^a(1+\sqrt{x})} = \frac{1}{x^a + x^{a+\frac{1}{2}}}$

$$\frac{1}{2x^{a+\frac{1}{2}}} < \frac{1}{x^a(1+\sqrt{x})} < \frac{1}{x^{a+\frac{1}{2}}} \text{ when } 1 \le x$$

hence, $\displaystyle\int_1^\infty \frac{1}{2x^{a+\frac{1}{2}}} < \int_1^\infty \frac{dx}{x^a(1+\sqrt{x})} < \int_1^\infty \frac{1}{x^{a+\frac{1}{2}}}$

$\displaystyle\int_1^\infty \frac{1}{x^{a+\frac{1}{2}}}$ converges if and only if $a > \dfrac{1}{2}$

hence, $\displaystyle\int_1^\infty \frac{dx}{x^a(1+\sqrt{x})}$ converges if and only if $a > \dfrac{1}{2}$

(b) $\displaystyle\lim_{x\to\frac{\pi}{2}} \sec^2 x = \infty$ This is an improper integral.

$$\begin{aligned}
\int_0^\pi \sec^2 x dx &= \int_0^{\frac{\pi}{2}} \sec^2 x dx + \int_{\frac{\pi}{2}}^\pi \sec^2 x dx \\
&= \lim_{t\to\frac{\pi}{2}^-} \int_0^t \sec^2 x dx + \lim_{t\to\frac{\pi}{2}^+} \int_t^\pi \sec^2 x dx \\
&= \lim_{t\to\frac{\pi}{2}^-} \tan t + \lim_{t\to\frac{\pi}{2}^+} \tan t \\
&= \infty
\end{aligned}$$

$\displaystyle\int_0^\pi \sec^2 x dx$ diverges.

5. (12%) Find the volume of the solid obtained by rotating about the y-axis the region between x-axis and $y = \cos x$, $0 \le x \le \dfrac{\pi}{2}$.

Sol: Sol.(I) Cutting into disks:

$$\begin{aligned}
\text{Volume} &= \int_0^1 \pi(\cos^{-1} y)^2 dy && \text{let } x = \cos^{-1} y \\
&= \int_{\pi/2}^0 \pi x^2(-\sin x)dx \\
&= \int_0^{\pi/2} \pi x^2 \sin x dx && \text{integration by parts} \\
&= \underbrace{\pi(-x^2 \cos x)\Big|_0^{\pi/2}}_{=0} + \int_0^{\pi/2} \pi \cdot 2x \cos x dx && \text{integration by parts} \\
&= \underbrace{2\pi \sin x\Big|_0^{\pi/2}}_{=\pi^2} - \int_0^{\pi/2} 2\pi \sin x dx \\
&= \pi^2 + 2\pi \cos x\Big|_0^{\pi/2} \\
&= \pi^2 - 2\pi
\end{aligned}$$

Sol.(II) Cutting into cylindrical shells:

$$\begin{aligned}
\text{Volume} &= \int_0^{\pi/2} 2\pi x \cos x dx && \text{integration by parts} \\
&= \underbrace{2\pi x \sin x\Big|_0^{\pi/2}}_{=\pi^2} - \int_0^{\pi/2} 2\pi \sin x dx \\
&= \pi^2 + 2\pi \cos x\Big|_0^{\pi/2} \\
&= \pi^2 - 2\pi
\end{aligned}$$

6. (12%) Solve the initial-value problem $y' = \dfrac{(y-4)(2x+1)}{(x^2+1)}$, $y(0) = -1$.

Sol:

$$\frac{dy}{dx} = (y-4)\frac{2x+1}{x^2+1} \quad \text{is seprable}$$

$$\frac{dy}{y-4} = \frac{2x+1}{x^2+1}$$

$$\int \frac{dy}{y-4} = \int \frac{2x+1}{x^2+1}$$

$$ln|y-4| = ln(x^2+1) + tan^{-1}x + c, \quad c \in \mathbb{R}$$

$$y = 4 + A(x^2+1)e^{tan^{-1}x}, \quad A \in \mathbb{R}$$

$$y(0) = -1 = 4 + A \quad \Rightarrow \quad A = -5$$

$$y = 4 - 5(x^2+1)e^{tan^{-1}x}$$

7. (12%) Solve the differential equation $x(x+1)y' + y + (x+1)^2 \cos x = 0$, $x > 0$, with $y(\pi) = \pi + 1$.

   Sol:

   $y' + \dfrac{y}{x(x+1)} = -\dfrac{x+1}{x}\cos x$

   $I(x) = \exp^{\int \frac{1}{x(x+1)}dx} = \exp^{\ln|\frac{x}{x+1}|} = \dfrac{x}{x+1}$

   $\Rightarrow I(x)y' + \dfrac{1}{(x+1)^2}y = -\cos x$

   $\Rightarrow (I(x)y)' = -\cos x$

   $\Rightarrow \dfrac{xy}{x+1} = -\sin x + c \Rightarrow y = -\dfrac{x+1}{x}\sin x + c\dfrac{x+1}{x}$

   Since $y(\pi) = \pi + 1 \Rightarrow c = \pi$

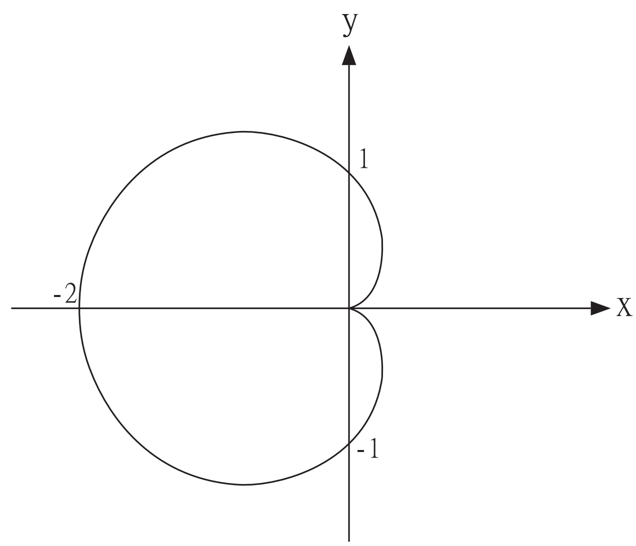
   $\Rightarrow y = \dfrac{x+1}{x}(\pi - \sin x)$

8. (14%) Consider the cardioid given by $r = 1 - \cos\theta$, $0 \le \theta \le 2\pi$.

   (a) Find the area enclosed by this curve.

   (b) Find the length of this curve.

   Sol:



(a)

$$
\begin{aligned}
Area &= \frac{1}{2}\int_0^{2\pi} r^2 d\theta \\
&= \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2 d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta d\theta \\
&= \frac{1}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]\Big|_0^{2\pi} \\
&= \frac{3}{2}\pi
\end{aligned}
$$

(b)

$$
\begin{aligned}
Length &= \int_0^{2\pi} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta \\
&= \int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} d\theta \\
&= \int_0^{2\pi} \sqrt{2 - 2\cos\theta} d\theta \\
&= \int_0^{2\pi} \sqrt{2(1 - \cos\theta)} d\theta \\
&= \int_0^{2\pi} \sqrt{2(2\sin^2(\frac{\theta}{2}))} d\theta \\
&= \int_0^{2\pi} 2\sin(\frac{\theta}{2}) d\theta \\
&= \Big[-4\cos(\frac{\theta}{2}) \Big] \Big|_0^{2\pi} = 8
\end{aligned}
$$