微甲01-05班 統一教學期中考解答

1. (16%, 每小題8分)

(a) Let $y = 2^{\tan^{-1} x} + (\ln x)^{\sqrt{x}}$. Find $\dfrac{dy}{dx}$.

(b) If $2y \sec x = 3x \tan y$, find $\dfrac{dy}{dx}$ at the point $(\frac{\pi}{3}, \frac{\pi}{4})$.

Sol:

(a) Since

$$y = 2^{\tan^{-1} x} + (\ln x)^{\sqrt{x}} = e^{\ln 2 \cdot \tan^{-1} x} + e^{\sqrt{x} \cdot \ln \ln x},$$

we have

$$\begin{aligned} \frac{dy}{dx} &= e^{\ln 2 \cdot \tan^{-1} x} \cdot \ln 2 \cdot \frac{1}{1+x^2} + e^{\sqrt{x} \cdot \ln \ln x} (\frac{1}{2\sqrt{x}} \cdot \ln \ln x + \sqrt{x} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}) \\ &= 2^{\tan^{-1} x} \cdot \ln 2 \cdot \frac{1}{1+x^2} + (\ln x)^{\sqrt{x}} (\frac{1}{2\sqrt{x}} \cdot \ln \ln x + \frac{1}{\ln x} \cdot \frac{1}{\sqrt{x}}). \end{aligned}$$

(b) Differentiate with respect to x,we have

$$2 \cdot \frac{dy}{dx} \cdot \sec x + 2y \sec x \tan x = 3 \tan y + 3x \sec^2 y \cdot \frac{dy}{dx}.$$

Hence, at $(x, y) = (\frac{\pi}{3}, \frac{\pi}{4})$, we have

$$2 \cdot \frac{dy}{dx} \cdot 2 + 2 \cdot \frac{\pi}{4} \cdot 2 \cdot \sqrt{3} = 3 \cdot 1 + 3 \cdot \frac{\pi}{3} \cdot (\sqrt{2})^2 \cdot \frac{dy}{dx}.$$

Therefore

$$\left.\frac{dx}{dy}\right|_{(\frac{\pi}{3}, \frac{\pi}{4})} = \frac{3 - \sqrt{3}\pi}{4 - 2\pi}.$$

2. (12%) Let

$$f(x) = \begin{cases} x^2 \sin \dfrac{\pi^2}{x} & \text{for } |x| \leq \pi,\ x \neq 0, \\ 0 & x = 0, \\ e^{\frac{x}{|x|-\pi}} & \text{for } |x| > \pi. \end{cases}$$

(a) At what values of x is $f(x)$ continuous?

(b) At what values of x is $f(x)$ differentiable?

Sol:

(a) $\lim_{x \to 0} |x^2 \sin \frac{\pi^2}{x}| \leq \lim_{x \to 0} |x^2| = 0 \Rightarrow \lim_{x \to 0} x^2 \sin \frac{\pi^2}{x} = 0 = f(0)$

$\Rightarrow f$ is continuous at $x = 0$.

$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^-} x^2 \sin \frac{\pi^2}{x} = \pi^2 \sin \pi = 0 = f(\pi)$

$\lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^+} e^{\frac{x}{x-\pi}} = +\infty \Rightarrow f$ is discontinuous at $x = \pi$.

$\lim_{x \to -\pi^+} f(x) = \lim_{x \to -\pi^+} x^2 \sin \frac{\pi^2}{x} = (\pi)^2 \sin \frac{\pi^2}{-\pi} = 0 = f(-\pi)$

$\lim_{x \to -\pi^-} f(x) = \lim_{x \to -\pi^-} e^{\frac{x}{-x-\pi}} = 0 = f(-\pi)$

$\Rightarrow f$ is continuous at $x = -\pi$. So f is continuous on $(-\infty, \pi) \cup (\pi, \infty)$

(b) Since f is discoutinuous at $x = \pi$, f is not differentiable at $x = \pi$.

$\lim_{x \to 0} |\frac{f(x) - f(0)}{x - 0}| = \lim_{x \to 0} |\frac{x^2 \sin \frac{\pi^2}{x}}{x}| = \lim_{x \to 0} |x \sin \frac{\pi^2}{x}| \leq \lim_{x \to 0} |x| = 0$

$\Rightarrow \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0 \Rightarrow f$ is differentiable at $x = 0$ and $f'(0) = 0$.

$\lim_{x \to -\pi^+} \frac{f(x) - f(-\pi)}{x - (-\pi)} = \lim_{x \to -\pi^+} \frac{x^2 \sin \frac{\pi^2}{x}}{x + \pi}$

$= \lim_{x \to -\pi^+} \frac{2x \sin \frac{\pi^2}{x} + x^2 (-\frac{\pi^2}{x^2}) \cos \frac{\pi^2}{x}}{1} = \pi^2$

$\lim_{x \to -\pi^-} \frac{f(x) - f(-\pi)}{x - (-\pi)} \lim_{x \to -\pi^-} \frac{e^{\frac{x}{-x-\pi}} - 0}{x + \pi} = \lim_{x \to -\pi^-} \frac{e^{-1} e^{\frac{\pi}{x+\pi}}}{x + \pi} = \lim_{t \to -\infty, t = \frac{1}{x+\pi}} e^{-1} t e^{\pi t}$

$\lim_{t \to -\infty} \frac{e^{-1} t}{e^{-\pi t}} = \lim_{t \to -\infty} \frac{e^{-1}}{-\pi e^{-\pi t}} = 0 \neq \pi^2 \Rightarrow f$ is not differentiable at $x = -\pi$.

So f is differentiable on $(-\infty, -\pi) \cup (-\pi, \pi) \cup (\pi, +\infty)$.

3. (32%, 每小題8分)

(a) Find $\lim_{x\to-\infty}(1+\frac{1}{x})^x$.

(b) Suppose $f(x)$ has a continuous second derivative $f''(x)$ for $x\in(a,b)$.
Find $\lim_{h\to 0}\frac{f(x+2h)-2f(x)+f(x-2h)}{h^2}$.

(c) Find $\lim_{n\to\infty}(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n})$.

(d) Find $\lim_{x\to 0}\frac{\int_0^x(\int_1^{\cos t}\sqrt{8+u^4}du)dt}{x^3}$.

Sol:

(a)

Sol.1 Let $y=(1+\frac{1}{x})^x \quad\Rightarrow\quad \ln y = x\cdot\ln(1+\frac{1}{x})$

Now, we can apply the L'Hospital Rule:

$$\begin{aligned}\lim_{x\to-\infty}\ln y &= \lim_{x\to-\infty} x\cdot\ln(1+\frac{1}{x})\\ &= \lim_{x\to-\infty}\frac{\ln(1+\frac{1}{x})}{(\frac{1}{x})}\\ &= \lim_{x\to-\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot(\frac{-1}{x^2})}{(\frac{-1}{x^2})}\\ &= \lim_{x\to-\infty}\frac{1}{1+(\frac{1}{x})}\\ &= 1\end{aligned}$$

Therefore,

$$\lim_{x\to-\infty}(1+\frac{1}{x})^x = \lim_{x\to-\infty} y = \lim_{x\to-\infty} e^{\ln y} = e^{\lim_{x\to-\infty}\ln y} = e^1 = e$$

since the exponential function is continuous on $\mathcal{R}$.

Q.E.D.

Sol.2 Let $y = -x \Rightarrow y \to \infty$ as $x \to -\infty$

$$\begin{aligned}
\lim_{x\to-\infty}(1+\frac{1}{x})^x &= \lim_{y\to\infty}(1-\frac{1}{y})^{-y} \\
&= \lim_{y\to\infty}(\frac{y-1}{y})^{-y} \\
&= \lim_{y\to\infty}(\frac{y}{y-1})^{y} \\
&= \lim_{y\to\infty}(1+\frac{1}{y-1})^{y} \\
&= \lim_{y\to\infty}\{(1+\frac{1}{y-1})^{y-1}\cdot(1+\frac{1}{y-1})\} \\
&= [\lim_{y\to\infty}(1+\frac{1}{y-1})^{y-1}]\cdot[\lim_{y\to\infty}(1+\frac{1}{y-1})] \\
&= [\lim_{u\to\infty}(1+\frac{1}{u})^{u}]\cdot[\lim_{u\to\infty}(1+\frac{1}{u})] \\
&= e\cdot 1 \\
&= e
\end{aligned}$$

Q.E.D.

(b)

Sol.1

$$\begin{aligned}
\lim_{h\to0}\frac{f(x+2h)-2f(x)+f(x-2h)}{h^2} &= \lim_{h\to0}\frac{[f(x+2h)-f(x)]-[f(x)-f(x-2h)]}{h^2} \\
&= \lim_{h\to0}\frac{2\cdot(\frac{f(x+2h)-f(x)}{2h})-2\cdot(\frac{f(x)-f(x-2h)}{2h})}{h} \\
&= 2\lim_{h\to0}2(\frac{f'(x)-f'(x-2h)}{2h}) \\
&= 4\lim_{h\to0}f''(x-2h) \\
&= 4f''(x)
\end{aligned}$$

Q.E.D.

Sol.2 Applying L'Hospital Rule,

$$
\begin{aligned}
\lim_{h\to 0}\frac{f(x+2h)-2f(x)+f(x-2h)}{h^2} &= \lim_{h\to 0}\frac{2f'(x+2h)-2f'(x-2h)}{2h} \\
&= \lim_{h\to 0}\frac{f'(x+2h)-f'(x-2h)}{h} \\
&= \lim_{h\to 0}[2f''(x+2h)+2f''(x-2h)] \\
&= 4f''(x)
\end{aligned}
$$

Q.E.D.

(c)

$$
\begin{aligned}
\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right) &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k} \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+(\frac{k}{n})} \\
&= \int_0^1 \frac{1}{1+x}dx \\
&= \ln|1+x|\ \Big|_0^1 \\
&= \ln 2-\ln 1 \\
&= \ln 2
\end{aligned}
$$

Q.E.D.

(d) By using L' Hospital's Rule twice, we get

$$
\begin{aligned}
& \lim_{x\to 0}\frac{\int_0^x\left(\int_1^{\cos t}\sqrt{8+u^4}\,du\right)\,dt}{x^3} \\
= \quad & \lim_{x\to 0}\frac{\int_1^{\cos x}\sqrt{8+u^4}\,du}{3x^2} \\
= \quad & \lim_{x\to 0}\frac{\sqrt{8+\cos^4 x}\cdot(-\sin x)}{6x} \\
= \quad & \lim_{x\to 0}\frac{-\sqrt{8+\cos^4 x}}{6}\cdot\frac{\sin x}{x} \\
= \quad & -\frac{1}{2}
\end{aligned}
$$

4. (8%) A number a is called a fixed point of a function $f(x)$ if $f(a) = a$. Prove that if $f(x)$ is differentiable and $f'(x) \neq 1$ for all real number x, then f has at most one fixed point.

Sol:

(Method I) Suppose $f(x)$ has two fixed points, called a and b, so $f(a) = a$, $f(b) = b$. Since $f(x)$ is differentiable for all real number, we use Mean Value Theorem and get:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some $c \in (a, b)$. Hence, $f'(c) = 1$ and it contradicts that $f'(x) \neq 1$ for all real number x. So $f(x)$ has at most one fixed point.

(Method II) Suppose $f(x)$ has two fixed points, called a and b, so $f(a) = a$, $f(b) = b$. Consider $g(x) = f(x) - x$. Because $f(x)$ is differentiable for all real number, $g(x)$ is also differentiable and $g'(x) = f'(x) - 1 \neq 0$ for all real number x from the hypothesis. Using Rolle's Theorem with $g(a) = 0$ and $g(b) = 0$, we get $g'(c) = 0$ for some $c \in (a, b)$, a contradiction.

5. (16%) Let $f(x) = \sin 2x + 4\sin x - x$, where $x \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$.

(a) Find the intervals of increasing and decreasing, local maximum values, local minimum values, intervals of concavity, and inflection points.

(b) Sketch the graph of $f(x)$.

Sol:

$f(x) = \sin 2x + 4\sin x - x, x \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$f'(x) = 2\cos 2x + 4\cos x - 1 = (2\cos x - 1)(2\cos x + 3)$

$f'(x) = 0 \Leftrightarrow x = -\frac{\pi}{3}, \frac{\pi}{3}, f(\frac{\pi}{3}) = \frac{5\sqrt{3}}{2} - \frac{\pi}{3}, f(-\frac{\pi}{3}) = -\frac{5\sqrt{3}}{2} + \frac{\pi}{3}$

$f''(x) = -4\sin 2x - 4\sin x = -4\sin x(2\cos x + 1)$

$f''(x) = 0 \Leftrightarrow x = 0, \pi, \frac{2}{3}\pi, \frac{4}{3}\pi$

$f(0) = 0, f(\pi) = -\pi, f(\frac{2}{3}\pi) = \frac{3}{2}\sqrt{3} - \frac{2}{3}\pi, f(\frac{4}{3}\pi) = -\frac{3}{2}\sqrt{3} - \frac{4}{3}\pi$

boundary points: $f(-\frac{\pi}{2}) = -4 + \frac{\pi}{2}, f(\frac{3}{2}\pi) = -4 - \frac{3}{2}\pi$

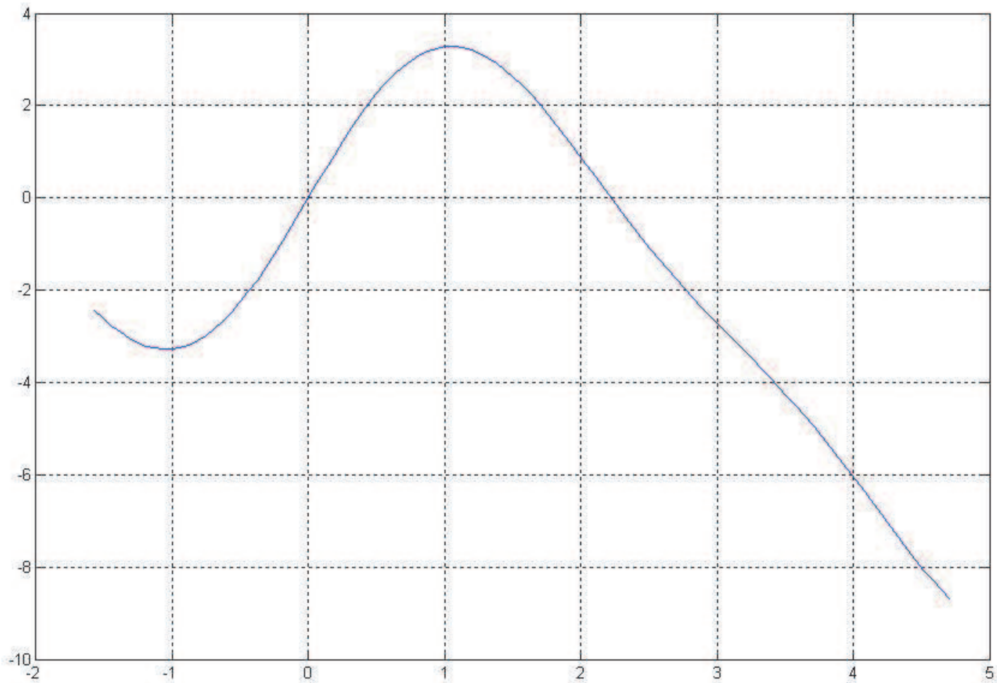
(a) increasing intervals: $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, decreasing intervals: $\left[\frac{\pi}{2}, -\frac{\pi}{3}\right), \left(\frac{\pi}{3}, \frac{3\pi}{2}\right]$

local maximum values: $\frac{5\sqrt{3}}{2} - \frac{\pi}{3}, -4 + \frac{\pi}{2}$, local minimum values: $-\frac{5\sqrt{3}}{2} + \frac{\pi}{3}, -4 - \frac{3}{2}\pi$

concave upward: $\left(-\frac{\pi}{2}, 0\right), \left(\frac{2}{3}\pi, \pi\right), \left(\frac{4}{3}\pi, \frac{3}{2}\pi\right)$, concave downward: $\left(0, \frac{2}{3}\pi\right), \left(\pi, \frac{4}{3}\pi\right)$

inflection points: $(0,0), (\pi, -\pi), \left(\frac{2}{3}\pi, \frac{3}{2}\sqrt{3} - \frac{2}{3}\pi\right), \left(\frac{4}{3}\pi, -\frac{3}{2}\sqrt{3} - \frac{4}{3}\pi\right)$

(b)



6. (8%) Suppose that the surface area of a right circular cylindrical solid is 1. Find the height and the radius of the solid when it is of maximal volume.

Sol:

Let the radius be r and height be h, $r, h > 0$. The surface area $A = 2\pi r^2 + 2\pi rh = 1$, thus $h = \frac{1-2\pi r^2}{2\pi r}$.

The volume $V = \pi r^2 h = \pi r^2 \dfrac{1-2\pi r^2}{2\pi r} = \dfrac{r - 2\pi r^2}{2} = V(r)$.

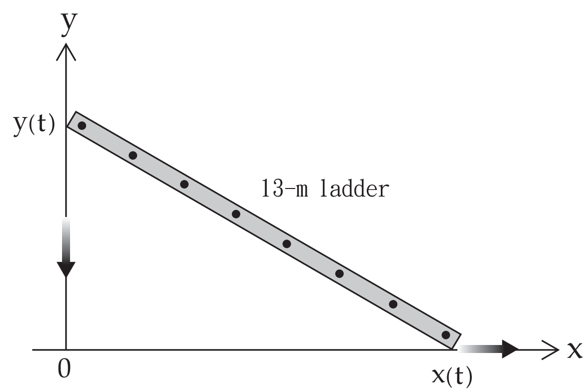
Set $\dfrac{d}{dr}V(r) = \dfrac{1-6\pi r^2}{2} = 0$, $r = \dfrac{\sqrt{6\pi}}{6\pi}$.

Since $\dfrac{d^2}{dr^2}V(r) = -6\pi r < 0$ on $(0, \frac{\sqrt{2\pi}}{2\pi})$ and $V(0) = V(\dfrac{\sqrt{2\pi}}{2\pi}) = 0$,

V has a unique local maximum (and hence maximum) at $r = \dfrac{\sqrt{6\pi}}{6\pi}$, $h = \dfrac{1 - 2\pi\frac{\sqrt{6\pi}}{6\pi}}{2\pi\frac{\sqrt{6\pi}}{6\pi}} = \dfrac{\sqrt{6\pi}}{3\pi}$.

7. (8%) A ladder 13m long is leaning against a wall when its base starts to slide away. By the time the base is 12m from the wall, the base is moving at the rate of 0.5 m/sec. At what rate is the area of the triangle formed by the ladder, the wall and the ground changing then?

Sol:



Let $A(t) = \frac{1}{2}x(t)y(t)$

To find $A'(t) = \frac{1}{2}(\frac{dx}{dt}y + \frac{dy}{dt}x)$

We have to derive $\frac{dy}{dt}$ from the equation $x^2 + y^2 = 13^2$

So $(x^2 + y^2)' = 0$ and $2x\frac{dx}{d}t + 2y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = -2x\frac{dx}{dt}\frac{1}{2y} = -2 \cdot 12 \cdot 0.5 \cdot \frac{1}{10} = -1.2$

Hence $A'(t) = \frac{1}{2}(0.5 \cdot 5 - 12 \cdot 1.2) = -5.95$ m^2/sec