微甲01-05班 統一教學期末考解答

1. (8%) Evaluate the integral $\int_{-1}^{1} \frac{e^{2x}}{1+e^x}\, dx$.

   Sol:

   Let $u = e^x$, then $du = e^x dx$, and so

$$\begin{aligned}\int_{-1}^{1} \frac{e^{2x}}{1+e^x} dx &= \int_{e^{-1}}^{e} \frac{u}{1+u} du \\ &= \int_{e^{-1}}^{e} 1 - \frac{1}{1+u} du \\ &= [u - \ln(1+u)]_{e^{-1}}^{e} \\ &= e - \frac{1}{e} - 1.\end{aligned}$$

2. (8%) Evaluate the integral $\int_{-1}^{1} \ln\left(x + \sqrt{1+x^2}\right) dx$.

   Sol:

   Note that for $-1 \le x \le 1$,

$$\ln(-x + \sqrt{1+(-x)^2}) = \ln \frac{1}{x + \sqrt{1+x^2}} = -\ln(x + \sqrt{1+x^2}).$$

   So $\ln(x + \sqrt{1+x^2})$ is an odd function in $[-1, 1]$, and thus the integral equals 0.

3. (8%) Evaluate the integral $\int \frac{dt}{t - \sqrt{1-t^2}}$.

   Sol:

   Set $t = \sin\theta$. Then the integral becomes to $\int \frac{\cos\theta}{\sin\theta - \cos\theta}\, d\theta$. Now

$$\int \frac{\cos\theta}{\sin\theta - \cos\theta}\, d\theta = \int \frac{1}{\tan\theta - 1}\, d\theta$$

   Again we make a substitution. Let $u = \tan\theta$. $du = \sec^2\theta d\theta = (1+u^2)d\theta$. Hence

$$\int \frac{1}{\tan\theta - 1}\, d\theta = \int \frac{1}{(u-1)(u^2+1)}\, du = \int \left(\frac{1}{2(u-1)} - \frac{u+1}{2(u^2+1)}\right) du$$

   Clearly,

$$\int \frac{1}{2(u-1)}\, du = \frac{1}{2} \ln|u-1| + C_1$$

$$\int \frac{u+1}{2(u^2+1)}\, du = \frac{1}{4}\ln|u^2+1| + \frac{1}{2}\arctan u + C_2$$

Hence one can get

$$\int \frac{1}{\tan\theta - 1}\, d\theta = \frac{1}{2}\ln|\tan\theta - 1| - \frac{1}{4}\ln|\tan^2\theta + 1| - \frac{1}{2}\theta + C$$

with $C = C_1 - C_2$. Note

$$\frac{1}{2}\ln|\tan\theta - 1| - \frac{1}{4}\ln|\tan^2\theta + 1| = \frac{1}{2}\ln|\tan\theta - 1| - \frac{1}{2}\ln|\sec\theta| = \frac{1}{2}\ln|\sin\theta - \cos\theta|$$

Recall that $\sin\theta = t$, $\cos\theta = \sqrt{1-t^2}$. Therefore

$$\int \frac{1}{t - \sqrt{1-t^2}}\, dt = \frac{1}{2}\ln|t - \sqrt{1-t^2}| - \frac{1}{2}\arcsin t + C$$

4. (8%) Evaluate the integral $\displaystyle\int \frac{dx}{\cos^2 x + a^2\sin^2 x}$ and $\displaystyle\int \frac{dx}{\cos^2 x - a^2\sin^2 x}$, $a > 0$.

Sol:

$$\int \frac{dx}{\cos^2(x) \pm a^2\sin^2(x)} = \int \frac{\sec^2(x)dx}{1 \pm a^2\tan^2(x)}$$

Let $t = \tan(x)$ (跟一般 $\tan(\frac{\theta}{2})$ 代換觀念一樣，這邊 sin，cos 都有平方，是兩倍角)

$$= \int \frac{dt}{1 \pm a^2t^2} = \frac{1}{a}\int \frac{du}{1 \pm u^2}, (u = at)$$

$$= \begin{cases} \dfrac{1}{a}\tan^{-1}(u) + C = \dfrac{1}{a}\tan^{-1}(a\tan(x)) + C, \text{ for } +a \\ \\ \dfrac{1}{a}\tanh^{-1}(u) + C = \dfrac{1}{a}\tanh^{-1}(a\tan(x)) + C, \text{ for } -a \end{cases}$$

$$\text{or } \frac{1}{2a}\ln\left|\frac{1+a\tan(x)}{1-a\tan(x)}\right| \text{ for } -a$$

5. (8%) Evaluate the improper integral $\displaystyle\int_0^\infty x^n e^{-x}\, dx$, n is a positive integer.

Sol:

Let $I_n = \displaystyle\int_0^\infty x^n e^{-x} dx = \lim_{t\to\infty} -x^n e^{-x}\Big|_0^t + n\int_0^\infty x^{n-1}e^{-x}$

$dx = nI_{n-1} \quad$ (since $\lim_{t\to\infty} x^n e^{-x}\Big|_0^t = 0$)

By Induction, $I_n = nI_{n-1} = n(n-1)I_{n-2} = \cdots = n!I_0$

Consider $I_0 = \int_0^\infty e^{-x}dx = \lim_{t\to\infty} -e^{-x}\Big|_0^t = 1$

Therefore, $I_n = n!I_0 = n!$

6. (8%) Evaluate the improper integral $\displaystyle\int_2^\infty \frac{4x^3+x-1}{x^2(x-1)(x^2+1)}\,dx.$

Sol:

$$f(x) \equiv \frac{4x^3+x-1}{x^2(x-1)(x^2+1)} = \frac{1}{x^2} + \frac{2}{x-1} + \frac{-2x+1}{x^2+1}$$

$$\int_2^\infty f(x)dx = \int_2^\infty \frac{1}{x^2} + \int_2^\infty \frac{2}{x-1} + \int_2^\infty \frac{-2x}{x^2+1} + \int_2^\infty \frac{1}{x^2+1}$$

$$= \lim_{t\to\infty}[-\frac{1}{t} + 2\ln|t-1| - \ln|t^2+1| + \tan^{-1} t] - [-\frac{1}{2} - \ln 5 + \tan^{-1} 2]$$

$$= \lim_{t\to\infty}[-\frac{1}{x} + \ln|\frac{(x-1)^2}{x^2+1}| + \tan^{-1} x] + \frac{1}{2} + \ln 5 - \tan^{-1} 2$$

$$= \frac{\pi}{2} + \frac{1}{2} + \ln 5 - \tan^{-1} 2$$

7. (8%) Solve the differential equation $(\cos x)\dfrac{dy}{dx} + (\sin x)y = \sin^2 x \cdot \tan^2 x.$

Sol:

$$\frac{dy}{dx} + \tan x \cdot y = \sin x \cdot \tan^3 x \cdots \bigotimes_1$$

$$I(x) = e^{\int \tan x dx} = e^{-ln|\cos x|} = \sec x$$

$$I(x) \cdot \bigotimes_1 = \sec x \cdot y' + \sec x \cdot \tan x \cdot y = \tan^4 x$$

$$\frac{d}{dx}\sec x \cdot y = \tan^4 x \Rightarrow \sec \cdot y = \int \tan^4 x dx$$

$$\because \int \tan^4 x = \int \tan^2 x(\sec^2 x - 1)dx = \int \tan^2 x d\tan x - \int \tan^2 x dx$$

$$= \frac{1}{3}\tan^3 x - \int(\sec^2 x - 1)dx = \frac{1}{3}\tan^3 x - \tan x + x + c$$

$$\therefore y = \frac{1}{3}\tan^2 x \sin x - \sin x + \cos x \cdot x + c\cos x$$

8. (8%) Find the orthogonal trajectories of the family of curves $y = \dfrac{k}{1+x^2}$. (Discuss for the cases $k = 0$ and $k \neq 0$.)

Sol:

case1($k \neq 0$):

$[y = \dfrac{k}{1+x^2} \Rightarrow y + x^2y = k \Rightarrow y' + 2xy + x^2y' = 0 \Rightarrow y' = \dfrac{-2xy}{1+x^2}]$

so the $\dfrac{dy}{dx}$ for the orthogonal trajectory is

$[\dfrac{dy}{dx} = \dfrac{1+x^2}{2xy} \Rightarrow \int 2y dy = \int \dfrac{1}{x} + x dx \Rightarrow y^2 = \ln|x| + \dfrac{x^2}{2} + C, y \neq 0]$

case2($k = 0$):

$\because k = 0 \therefore$ the function is $y = 0$(x-axis)

so the orthogonal trajectory is $x = c$, $c \in$R

9. (20%) A curve is defined by

$$\begin{cases} x = t^2 - 2t \\ y = -t^3 + 3t^2 \end{cases}, t \in \mathbb{R}.$$

(a) Find the point at which the curve crosses itself.

(b) Find the points on C where the tangent is horizontal or vertical.

(c) Let R be the region enclosed by the loop of the curve. If R is rotated about the line $x = -1$, find the volume of the resulting solid.

(d) Find the centroid of R.

Sol:

(a) Let $t_1 \neq t_2 \in$ R , and $(t_1{}^2 - 2t_1, -t_1{}^3 + 3t_1{}^2) = (t_2{}^2 - 2t_2, -t_2{}^3 + 3t_2{}^2)$

$$\Rightarrow \begin{cases} t_1{}^2 - 2t_1 = t_2{}^2 - 2t_2 \\ -t_1{}^3 + 3t_1{}^2 = -t_2{}^3 + 3t_2{}^2 \end{cases}$$

$$\Rightarrow \begin{cases} (t_1 - t_2)(t_1 + t_2) = 2(t_1 - t_2)) \\ 3(t_1 - t_2)(t_1 + t_2) = (t_1 - t_2)(t_1^2 + t_1 t_2 + t_2^2) \end{cases}$$

Since $t_1 \neq t_2 \Rightarrow \begin{cases} t_1+t_2= 2 \\ t_1t_2=-2 \end{cases}$

$\Rightarrow t_1,\ t_2=1 \pm \sqrt{3}$ So, we have the point (2,2) i.e. the intersection of the curve.

(b) (i) the point on C where the tangent line of horizontal

Let $\frac{dy}{dt}=0 \Rightarrow -3t^2+6t = 0 \Rightarrow t=0$ or 2 then we have the points (0,0) , (0,4).

(ii) the point on C where the tangent line of vertical (Sol) Let $\frac{dy}{dt}=0 \Rightarrow 2t-2 = 0 \Rightarrow t = 1$

then we have the point (-1,2).

(c)

$$\begin{aligned}
V &= \int 2\pi r(dr)h \\
&= 2\pi \int (x+1)(dx)2(y-2) \\
&= -8\pi \int_{t=1}^{t=1+\sqrt{3}} (t-1)^4[(t-1)^2-3]d(t-1) \\
&= -8\pi \int_{a=0}^{a=\sqrt{3}} a^4[a^2-3]da \\
&= \frac{432\sqrt{3}\pi}{35}
\end{aligned}$$

(d) $A = \int_{t=1}^{t=1+\sqrt{3}} -4(t^2-2t)(t-1)[(t-1)^2-3](t-1)dt = \frac{48\sqrt{3}}{35}$

$R = \int_{t=1}^{t=1+\sqrt{3}} -4(t-1)[(t-1)^2-3](t-1)\ d(t-1) = \frac{24\sqrt{3}}{5}$

$x_c = \frac{A}{R} = \frac{2}{7}, y_c = 2$

10. (16%) Let R be the region lying inside the curve $r^2 = 2\cos\theta$ and outside the curve $r = 2-2\cos\theta$.

(a) Find the area of R.

(b) Find the area of the surface of the solid obtained by rotating R about the x-axis.

Sol:

$2\cos\theta = (2-2\cos\theta)^2$

$\Rightarrow 4\cos^2\theta - 10\cos\theta + 4 = 0$

$\Rightarrow (2\cos\theta - 1)(\cos\theta - 2) = 0$

$\Rightarrow \theta = \pm\frac{\pi}{3}$, which is the angle where two curve intersect.

(a)

$$\begin{aligned} R &= 2\int_0^{\frac{\pi}{3}} \cos\theta - \frac{1}{2}(2-2\cos\theta)^2 d\theta \\ &= \int_0^{\frac{\pi}{3}} -4 + 10\cos\theta - 4\cos^2\theta d\theta \\ &= \int_0^{\frac{\pi}{3}} -6 + 10\cos\theta - 2\cos 2\theta d\theta \\ &= \frac{9\sqrt{3}}{2} - 2\pi \end{aligned}$$

(b)

$$\begin{aligned} A &= \int_0^{\frac{\pi}{3}} 2\pi y ds = \int_0^{\frac{\pi}{3}} 2\pi r\sin\theta\sqrt{\dot{r}^2 + r^2} d\theta \\ &= \int_0^{\frac{\pi}{3}} 2\pi\sqrt{2\cos\theta}\sin\theta\sqrt{\frac{\sin^2\theta}{2\cos\theta} + 2\cos\theta} d\theta + \int_0^{\frac{\pi}{3}} 2\pi(2-2\cos\theta)\sin\theta\sqrt{(2\sin\theta)^2 + (2-2\cos\theta)^2} d\theta \\ &= -2\pi\int_0^{\frac{\pi}{3}} \sqrt{1+3\cos^2\theta} d\cos\theta + 2\pi\int_0^{\frac{\pi}{3}} (2-2\cos\theta)^{\frac{3}{2}} d(2-2\cos\theta) \\ &= \frac{2\pi}{\sqrt{3}}\int_{\tan\theta=\frac{\sqrt{3}}{2}}^{\tan\theta=\sqrt{3}} \sec^3\theta d\theta + \frac{4}{5}\pi \\ &= \frac{2\pi}{\sqrt{3}}\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right)\Big|_{\tan\theta=\frac{\sqrt{3}}{2}}^{\tan\theta=\sqrt{3}} + \frac{4}{5}\pi \\ &= \frac{14\pi}{5} - \frac{\sqrt{7}\pi}{4} + \frac{\pi}{\sqrt{3}}\ln\frac{4+\sqrt{3}}{\sqrt{7}+\sqrt{3}} \end{aligned}$$