962　微積分甲　08-12班　期末考題

1. (10%) 令 R 為矩形 $-1 \le x \le 1,\ 0 \le y \le 2$。求 $\iint_R |y-x^2|dA$ 。

$$\begin{aligned}\iint_R |y-x^2|dA &= \int_{-1}^{1}\int_0^{x^2}(x^2-y)dydx + \int_{-1}^{1}\int_{x^2}^{2}(y-x^2)dydx \quad \text{(5 points)}\\ &= \int_{-1}^{1}(tx^2-\frac{1}{2}x^2y)|_0^{x^2}dx + \int_{-1}^{1}(\frac{1}{2}y^2-x^2y)|_{x^2}^{2}dx \quad \text{(2 points)}\\ &= \frac{46}{15} \quad \text{(1 point)}\end{aligned}$$

2. (10%) 求 $\int_0^{\frac{1}{2}}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} \ln(x^2+y^2)dxdy$ 。

$$\begin{aligned}\int_0^{1/2}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} \ln(x^2+y^2)dxdy &= \int_0^{\frac{\pi}{6}}\int_0^1 \ln r^2\cdot rdrd\theta \quad \text{(4 points)}\\ &= \frac{\pi}{6}\cdot\lim_{a\to 0}\frac{1}{2}(r^2\ln r^2-r^2)|_a^1\\ &= -\frac{\pi}{12} \quad \text{(2 points)}\end{aligned}$$

3. (10%) 求 $\int_0^2\int_0^{4-x^2}\int_0^x \frac{\sin 2z}{4-z}dydzdx$ 。

$$\begin{aligned}\text{原式} &= \int_0^2\int_0^{4-x^2}\frac{\sin 2z}{4-z}xdzdx \quad \text{(2 points)}\\ &= \int_0^4\int_0^{\sqrt{4-z}}\frac{\sin 2z}{4-z}xdxdz \quad \text{(2 points)}\end{aligned}$$

之後的兩個積分共五分, 上下限代錯扣一分

4. (12%) 令 $D=\{(x,y,z)|2x^2+3y^2+5z^2+6yz+2xz\le 1\}$。求 $\iiint_D (x+y+z)^2 dV$ 。
(提示: 利用 $2x^2+3y^2+5z^2+6yz+2xz=(x+y+z)^2+(x-y)^2+(y+2z)^2$, 可以將 D 轉換成一球體。)

$$\text{Let}\begin{cases} u &= x+y+z \\ v &= x-y \\ w &= y+2z \end{cases} \text{then}\begin{cases} x &= \frac{1}{3}(2u+v-w) \\ y &= \frac{1}{3}(2u-2v-w) \\ z &= \frac{1}{3}(-u+v+2w) \end{cases}$$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|\begin{vmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{vmatrix}\right| = \left|-\frac{1}{3}\right| = \frac{1}{3}$$

$$\iiint_D (x+y+z)^2 dV = \frac{1}{3}\iiint_D u^2 du dv dw$$

$$\text{Let}\begin{cases} u &= \rho\sin\phi\cos\theta \\ v &= \rho\sin\phi\sin\theta \\ w &= \rho\cos\phi \end{cases} \text{then}\left|\frac{\partial(u,v,w)}{\partial(\rho,\phi,\theta)}\right| = \rho^2\sin\phi$$

$$\begin{aligned} \frac{1}{3}\iiint_D u^2 du dv dw &= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\int_0^{1} (\rho\sin\phi\cos\theta)^2\cdot\rho^2\sin\phi\, d\rho d\phi d\theta \\ &= \frac{4}{45}\pi \end{aligned}$$

5. (14%) 令 $\mathbf{F} = yz^2\mathbf{i} + (xz^2 + ze^{yz})\mathbf{j} + (2xyz + p(y,z) + \dfrac{1}{1+z})\mathbf{k}$, 其中 $p(y,z)$ 爲變數 y, z 的函數, 它有連續的一階偏導函數, 且 $p(0,z) = 0$ 。假設 $\mathbf{F}$ 爲保守場 (conservative):

(a) 求 $p(y,z)$ 。

(b) 求 $\mathbf{F}$ 的位勢函數 (potential function)。

(c) 令曲線 C 爲 $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \le t \le 1$, 求 $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ 。

ϕ is a potential,

$\frac{\partial \phi}{\partial x} = yz^2 \Rightarrow \phi = xyz^2 + \theta(y,z)$

$\frac{\partial \phi}{\partial y} = xz^2 + \frac{\partial \theta}{\partial y}(y,z) = xz^2 + ze^{yz}$

$\frac{\partial \theta}{\partial y} = ze^{yz} \Rightarrow \theta = e^{yz} + \sigma(z)$

and

$\phi = xyz^2 + e^{yz} + \sigma(z)$

$\frac{\partial \phi}{\partial z} = 2xyz + ye^{yz} + \sigma'(z) = 2xyz + p(y,z) + \frac{1}{1+z}$

$\Rightarrow ye^{yz} = p(y,z), \sigma(z) = \ln(1+z) + C$

(a) $p(y,z) = ye^{yz}$

(b) $\phi = xyz^2 + e^{yz} + \ln(1+z) + C$

(c) $\int_C \vec{F} dr = \phi(1,1,1) - \phi(0,0,0) = e + \ln 2$

6. (10%) 令 R 爲矩形 $1 \le x \le \sqrt{3}, \frac{1}{\sqrt{3}} \le y \le 1$, C 爲 R 的邊界, 且取逆時針方向, $\mathbf{n}$ 爲往外的單位法向量; $\mathbf{F} = (\frac{x+2y\tan^{-1}x}{1+y^2})\mathbf{i} + (\frac{y-\ln(1+y^2)}{1+x^2})\mathbf{j}$。求 $\oint_C \mathbf{F}\cdot\mathbf{n}ds$ 。

$$div\vec{F} = \frac{1}{1+y^2} + \frac{2y}{(1+x^2)(1+y^2)} + \frac{1}{1+x^2} - \frac{2y}{(1+x^2)(1+y^2)} = \frac{1}{1+y^2} + \frac{1}{1+x^2}$$

Use Green's Thm

$$\begin{aligned}\oint_C = \mathbf{F}\cdot\mathbf{n}ds &= \iint_R div\mathbf{F}dA \\ &= \int_1^{\sqrt{3}}\int_{\frac{1}{\sqrt{3}}}^1 \frac{1}{(1+y^2)} + \frac{1}{1+x^2}dydx \\ &= \frac{3}{18}\pi\end{aligned}$$

也可以在每個邊上運用線積分去計算, 也可以得到相同結果。

$$\oint_C \mathbf{F}\cdot\mathbf{n}ds = \int_{\frac{1}{\sqrt{3}}}^1 \mathbf{F}(\sqrt{3},y)\cdot(1,0)dy + \int_1^{\sqrt{3}} \mathbf{F}(x,1)\cdot(0,1)dx + \int_{\frac{1}{\sqrt{3}}}^1 \mathbf{F}(1,y)\cdot(-1,0)dy + \int_1^{\sqrt{3}} \mathbf{F}(x,\frac{1}{\sqrt{3}})\cdot(0,-1)dx$$

其中 y 的部分

$$\int_{\frac{1}{\sqrt{3}}}^1 \frac{\sqrt{3}+\frac{\pi}{3}2y}{1+y^2}dy - \int_{\frac{1}{\sqrt{3}}}^1 \frac{1+\frac{\pi}{4}2y}{1+y^2}dy = \frac{\pi}{12}(\sqrt{3}-1) + \frac{\pi}{12}(\ln 2 - \ln\frac{4}{3})$$

x 的部分

$$\int_1^{\sqrt{3}} \frac{1-\ln 2}{1+x^2}dx - \int_1^{\sqrt{3}} \frac{\frac{1}{\sqrt{3}} - \ln\frac{4}{3}}{1+x^2} = \frac{\pi}{12}(1-\frac{1}{\sqrt{3}}) - \frac{\pi}{12}(\ln 2 - \ln\frac{4}{3})$$

7. (12%) 令 S 爲曲面 $z = x^2 + y^2$ 被平面 $z = 0$ 及 $z = 1$ 所截出的部份; 單位法向量 $\mathbf{n}$ 的方向指離 z 軸 (即 $-\mathbf{n}$ 指向 z 軸); $\mathbf{F} = 4x\mathbf{i} + 4y\mathbf{j} + 2\mathbf{k}$。求 $\mathbf{F}$ 經由 S 的外通量 (outward flux)。

法向量 $\vec{n} = (2x, 2y, -1)/\sqrt{1 + 4(x^2 + y^2)}$, $d\sigma = \sqrt{1 + 4(x^2 + y^2)}dxdy \Rightarrow$(6 points)

$\mathbf{r}_r \times \mathbf{r}_\theta = (-2r^2 \cos\theta, -2r^2 \sin\theta, r)$ (3 points)

$\nabla \cdot F = 8 \Rightarrow$(1point)

$\iiint \nabla \cdot F dV = 4\pi \Rightarrow$(1point)

圓錐, 但算對 (6points)

div.thm – 上蓋, 算錯 (6 8 points)

計算錯誤 (8 12 points)

$\nabla \times F$ (0point) , $\nabla \cdot \mathbf{F} = (4, 4, 2)$,etc. (0point)

8. (10%) 令 S 爲柱面 $4x^2+9y^2=36, 0\le z\le 5$，在 $z=5$ 處加上頂蓋 $4x^2+9y^2\le 36$; 單位法向量 $\mathbf{n}$ 的方向指離 z 軸 (即 $-\mathbf{n}$ 指向 z 軸); $\mathbf{F}=-y\mathbf{i}+(x+z)\mathbf{j}+x^2\mathbf{k}$。求 $\nabla\times\mathbf{F}$ 經由 S 的外 通量 (outward flux)。

S 爲整個柱面 (包含頂蓋),S_1爲側面,S_2爲頂蓋,S_3爲底部。其中 $S=S_1\cup S_2$ 且 $C=\partial S_3$, 爲逆時針 (由於法向量指向柱體外部)。

Solution1

$C:\{x=3\cos t, y=2\sin t, z=0, 0\le t\le 2\pi\}$ By Stokes' thm

$$\iint_S(\nabla\times\mathbf{F})\cdot\vec{n}d\sigma=\oint_C\mathbf{F}\cdot dt=\int_0^{2\pi}(-2\sin t, 3\cos t, 9\cos^2 t)\cdot(-3\sin t, 2\cos t, 0)dt=12\pi$$

Solution2

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ -y&x+z&x^2\end{vmatrix}=-\mathbf{i}-2x\mathbf{j}+2\mathbf{k}$$

根據 Divergence Thm

$$\iint_S(\nabla\times\mathbf{F})\cdot\vec{n}d\sigma+\iint_{S_3}(\nabla\times\mathbf{F})\cdot\vec{n}d\sigma=\iiint_V\nabla\cdot(\nabla\times\mathbf{F})dV=0$$

$$\iint_{S_3}(\nabla\times\mathbf{F})\cdot\vec{n}d\sigma=\iint_{S_3}(-1,-2x,2)\cdot(0,0,-1)dA=-12\pi \qquad \therefore\iint_S(\nabla\times\mathbf{F})\cdot\vec{n}d\sigma=12\pi$$

註: 底的面積亦可由 Green 定理求出

Solution3

$S_1:\{x=3\cos s, y=2\sin s, z=t, 0\le s\le 2\pi, 0\le t\le 5\}\Rightarrow r_1(s,t)=(3\cos s, 2\sin s, t)$

$\frac{\partial}{\partial s}r_1=(-3\sin s, 2\cos s, 0)$, $\frac{\partial}{\partial t}r_1=(0,0,1)$

$\vec{n}_{S_1}=(-3\sin s, 2\cos s, 0)\times(0,0,1)=(2\cos s, 3\cos s, 0)$

$\iint_{S_1}(\nabla\times\mathbf{F})\cdot\vec{n}d\sigma=\int_0^5\int_0^{2\pi}(-1,-2\cdot 3\cos s, 2)\cdot(2\cos s, 3\sin s, 0)dsdt=0$

$S_2:\{x=3r\cos t, y=2r\sin t, z=5, 0\le t\le 2\pi, 0\le r\le 1\}$

Similarily, $\iint_{S_2}(\nabla\times\mathbf{F})\cdot\vec{n}d\sigma=12\pi$

9. (12%) 令 S 爲環狀柱體 $1 \le x^2 + y^2 \le 2, 1 \le z \le 2$ 的全表面; 單位法向量 $\mathbf{n}$ 的方向 指離 z 軸 (即 $-\mathbf{n}$ 指向 z 軸); $\mathbf{F} = \ln(x^2 + y^2)\mathbf{i} - xyz\mathbf{j} + z^2\sqrt{x^2 + y^2}\mathbf{k}$。求 $\mathbf{F}$ 經由 S 的外通量 (outward flux)。

$$
\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma &= \iint_V div\mathbf{F} dV \\
&= \iiint_V \frac{2x}{x^2 + y^2} - xz + 2z\sqrt{x^2 + y^2} dV \\
&= \int_0^{2\pi} \int_1^{\sqrt{2}} \int_1^2 r\left(\frac{2r\cos\theta}{r^2} - rz\cos\theta + 2rz\right) dz dr d\theta \\
&= 2\pi(2\sqrt{2} - 1)
\end{aligned}
$$

亦可不用 Divergence Theorem, 分成上下內外四部分, 直接計算

外:$\iint_S (\nabla \times \mathbf{F}) \cdot \vec{n} d\sigma = \int_1^2 \int_0^{2\pi} (\sqrt{2}\cos\theta \ln 2 - 2\sqrt{2}\cos\theta \sin^2\theta z)\sqrt{2} d\theta dz = \cdots = 0$

內:$\iint_S (\nabla \times \mathbf{F}) \cdot \vec{n} d\sigma = \int_1^2 \int_0^{2\pi} (\cos\theta \sin^2\theta z) d\theta dz = \cdots = 0$

上:$\iint_S (\nabla \times \mathbf{F}) \cdot \vec{n} d\sigma = \int_0^{2\pi} \int_1^2 (4r^2) dr d\theta = \cdots = \frac{8}{3}\pi(2\sqrt{2} - 1)$

下:$\iint_S (\nabla \times \mathbf{F}) \cdot \vec{n} d\sigma = \int_0^{2\pi} \int_1^2 (-r^2) dr d\theta = \cdots = \frac{-2}{3}\pi(2\sqrt{2} - 1)$