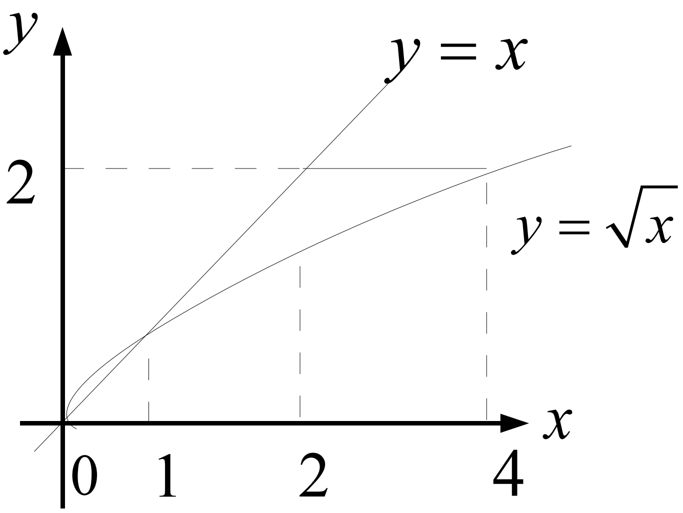
1. (15%) (a) (7%) Sketch the region of integration and evaluate the integral

$$\int_1^2\int_{\sqrt{x}}^{x} \frac{\sin y}{y}\, dy\, dx + \int_2^4\int_{\sqrt{x}}^{2} \frac{\sin y}{y}\, dy\, dx.$$

(b) (8%) Express and compute the moment of inertia I_z of the solid hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, in spherical coordinates. Take density function $\delta(x, y, z) = 1$.

(a)



(2 points)

$$\begin{aligned}\int_1^2 \int_{\sqrt{x}}^{x} \frac{\sin y}{y} dydx + \int_2^4 \int_{\sqrt{x}}^{2} \frac{\sin y}{y} dydx &= \int_1^2 \int_{\sqrt{y}}^{y^2} \frac{\sin y}{y} dxdy \quad \text{(4 points)} \\ &= \sin 2 - \cos 2 - \sin 1. \quad \text{(1 points)}\end{aligned}$$

(b)

$$\begin{aligned}&\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^2 (\rho \sin\phi)^2 \rho^2 \sin\phi d\rho d\phi d\theta \quad \text{(7 points)} \\ = &\ \frac{128}{15}\pi \quad \text{(1 points)}\end{aligned}$$

2. (10%) $\displaystyle\iint_R (x+y)e^{x^2-y^2}dA$, where R is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$.

Let $u=x-y, v=x+y$, then $x=\frac{u+v}{2}, y=\frac{u-v}{2}$ (2 points)

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2} \quad \text{(2 points)}$$

$$\int\int_R (x+y)e^{x^2-y^2}\,dA = \int_0^3\int_0^2 ue^{uv}\left|-\frac{1}{2}\right|dvdu \quad \text{(3 points)}$$

$$= \frac{1}{2}\int_0^3 (e^{2v}-1)du = \frac{1}{4}(e^6-7) \quad \text{(3 points)}$$

沒用 Jacobian (+0 2)

用 $\frac{\partial(x,y)}{\partial(u,v)}$ 當 Jacobian (+0 4)

算 Jacobian 可是積分中沒用 (+0 6)

積分中 Jacobian 沒絕對值 (+0 8)

3. (10%) Find the circulation of the field $\mathbf{F} = -x^2y\,\mathbf{i} + xy^2\,\mathbf{j} + z^2\,\mathbf{k}$ around the curve C in which the circular cylinder $x^2+y^2=2x$ meets the cone $z=2\sqrt{x^2+y^2}$, counterclockwise as viewed from above.

SOL1

Let $g(x,y,z) = z - 2\sqrt{x^2+y^2}$(or $z^2-4(x^2-y^2)$), (1 point)

then $\nabla g = (\frac{-2x}{\sqrt{x^2+y^2}}, \frac{-2y}{\sqrt{x^2+y^2}}, 1)$ (or $(-8x,-8y,2z)$) (1 point)

and $\nabla\times\mathbf{F} = (x^2+y^2)\mathbf{k}$ (1 point)

$$\begin{aligned}
\oint_C \mathbf{F}\cdot dt &= \int\int_S \nabla\times\mathbf{F}\cdot\vec{n}d\sigma \quad \text{(by Stoke's thm.)} \\
&= \int\int_{\{(x-1)^2+y^2\le 1\}} (x^2+y^2)\mathbf{k}\cdot\frac{\nabla g}{|\nabla g|}\frac{|\nabla g|}{|\nabla g\cdot k|}dxdy \quad \text{(2 points)} \\
&= \int\int_{\{(x-1)^2+y^2\le 1\}} (x^2+y^2)dxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^3 drd\theta \quad \text{(3 points)} \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(2\cos\theta)^4}{4}d\theta = \frac{3\pi}{2} \quad \text{(2 points)}
\end{aligned}$$

SOL2

$x^2+y^2=2x \Rightarrow (x-1)^2+y^2=1 \Rightarrow x=\cos\theta+1, y=\sin\theta \Rightarrow z=2\sqrt{x^2+y^2}=2\sqrt{2(\cos\theta+1)}$

Let $C(\theta)=(\cos\theta+1,\sin\theta,2\sqrt{2(\cos\theta+1)})$ (3 points 看參數化對不對)

$$\int_{C(\theta)} \mathbf{F}\cdot dT = \int_0^{2\pi} (\cos\theta+1,\sin\theta,2\sqrt{2(\cos\theta+1)})\cdot(-\sin\theta,\cos\theta,\frac{-2\sin\theta}{\sqrt{2(\cos\theta+1)}})d\theta \quad \text{(3points)}$$

$$= \int_0^{2\pi} \left((\cos\theta+1)^2\sin^2\theta+(\cos\theta+1)\cos\theta\sin^2\theta-8\sqrt{2(\cos\theta)+1}\sin\theta\right)d\theta$$

$$= \frac{3}{2}\pi \quad \text{(4 points)}$$

4. (10%) Evaluate the line integral $\int_C (ye^{xy}+3x^2y)dx+(xe^{xy}+x^3+1)dy$, where C is the circle $x^2+y^2=1$ connecting the points $A=(1,0)$ and $B=(0,1)$. Orientation from B to A.

Let $F(x,y)=(M(x,y),N(x,y))=(ye^{xy}+3x^2y,xe^{xy}+x^3+1)$

$$\frac{\partial M}{\partial y}=e^{xy}+xye^{xy}+3x^2, \frac{\partial N}{\partial x}=e^{xy}+xye^{xy}+3x^2 \quad \text{(3 points)}$$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \Rightarrow F(x,y) \text{ is conservative .}$$

$$\therefore \text{there exists a function } f:\mathbb{R}^2\to\mathbb{R}\text{such that}\nabla f=F. \quad \text{(1 point)}$$

$$\frac{\partial f}{\partial x}=ye^{xy}+3x^2y \Rightarrow f(x,y)=e^{xy}+x^3y+g(y) \quad \text{(2 points)}$$

$$\frac{\partial f}{\partial y}=xe^{xy}+x^3+g'(y)=xe^{xy}+x^3+1 \Rightarrow g'(y)=1 \Rightarrow g(y)=y \quad \text{(2 points)}$$

$$\therefore f(x,y)=e^{xy}+x^3y+y$$

$$\text{原式}=\int_B^A \nabla f\cdot dr=f(A)-f(B)=f(1,0)-f(0,1)=(1+0+0)-(1+0+1)=-1$$

5. (25%) A piece of surface S is cut out from $z=xy+1$ by $x^2+y^2=1$, and given a vector field $\mathbf{F}(x,y,z)=(xz+y,\ y-3x,\ x^2)$

(a) (5%) Compute surface area of S.

(b) Suppose ∂S is the boundary of S and is oriented counterclockwise when viewed from above.

(i) (5%) Compute the line integral $\oint_{\partial S} \mathbf{F}\cdot d\mathbf{r}$ directly.

(ii) (5%) Compute the line integral $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$ by any theorem.

(c) (5%) Surface $z = xy + 1$, $x^2 + y^2 = 1$ and xy plane enclose a solid region V, find the total flux of $\mathbf{F}$ across V.

(d) (5%) (i) Find the total flux of curl $\mathbf{F}$ across V.

(ii) Find the total flux of curl $\mathbf{G}$ across V, for $\mathbf{G}$ is any other given smooth vector field in $\mathbb{R}^3$.

$\phi(x, y, z) = (x, y, xy + 1)$

(a) $\phi_x = (1, 0, y) \quad \phi_x \times \phi_y = (-y, -x, 1) \quad \phi_y = (0, 1, x)$

$A = \int\int_D |\phi_x \times \phi_y| dxdy = \int\int_R \frac{\sqrt{y^2+x^2+1}}{1} dA$ (2 points)

$= \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1} r dr d\theta = 2\pi(1 + r^{3/2})|_0^1 = \frac{2\pi}{3}(2\sqrt{2} - 1)$ (3 pionts)

(b) (1)

$r(\theta) = (\cos\theta, \sin\theta, \cos\theta \sin\theta + 1)$

$F(\theta) = (\cos^2\theta \sin\theta + \cos\theta + \sin\theta, \sin\theta - 3\cos\theta, \cos^2\theta)$ (1point)

$r'(\theta) = (-\sin\theta, \cos\theta, \cos^2\theta - \sin^2\theta)$(1point)

$$\begin{aligned}
\oint \mathbf{F} dr &= \int_0^{2\pi} \left[-\sin^2\theta\cos^2\theta - \sin\theta\cos\theta - \sin^2\theta + \sin\theta\cos\theta - 3\cos^2\theta + \cos^4\theta - \cos^2\theta\sin^2\theta\right] d\theta \\
&= \int_0^{2\pi} -\frac{1}{2}\sin^2 2\theta - \frac{1-\cos 2\theta}{2} - 3\frac{1+\cos 2\theta}{2} + \frac{(1+\cos 2\theta)^2}{4} d\theta \\
&= -\frac{15}{4}\pi \quad \text{(3 points)}
\end{aligned}$$

(2)

$\nabla \mathbf{F} = (0, -x, -4)$ (1 point)

$$\begin{aligned}
\oint \mathbf{F} dr &= \int\int_S \nabla \times \mathbf{F} \cdot \vec{n} d\sigma = \int\int_D (x^2 - 4) dxdy \quad \text{(1 point)} \\
&= \int_0^{2\pi} \int_0^1 (r^2 \cos^2\theta - 4) r dr d\theta \\
&= \int_0^{2\pi} \left(\frac{1}{4}\cos^2\theta - 2\right) d\theta \\
&= \frac{1}{4}\pi - 2 \cdot 2\pi = -\frac{15}{4}\pi \quad \text{(3 points)}
\end{aligned}$$

(c)

$$\begin{aligned} \int\int_{\partial V} \mathbf{F}\cdot\vec{n}dA &= \int\int\int_V div\mathbf{F}d\sigma \\ &= \int\int\int_V (z+1)d\sigma = \int_0^{2\pi}\int_0^1\int_0^{r^2\cos\theta\sin\theta+1} (z+1)dzrdrd\theta \\ &= \frac{73}{48}\pi \quad \text{(3 points)} \end{aligned}$$

(d)

(1) $\int\int_{\partial V} \nabla\times\mathbf{F}\cdot\vec{n}dS = \oint_\phi \mathbf{F}\cdot dr = 0$ (2 points)

(2) $\int\int_{\partial V} \nabla\times G\cdot\vec{n} = \int\int\int_V div(\nabla\times G)dS = 0$ (3 points)

6. (15%) Find the outward flux of the vector field $\mathbf{V}(x,y,z) = (\frac{x}{r^3} + 2y + 3z)\mathbf{i} + (\frac{y}{r^3} + x + 3z)\mathbf{j} + (\frac{z}{r^3} + x + 2y)\mathbf{k}$ across the boundary of the ellipsoid region $D = 11x^2 + 12y^2 + 13z^2 \le 14$ where $r = \sqrt{x^2+y^2+z^2}$. (No points without complete computation and explanation.)

V is not differentiable in D since $v = 0$ at $(x,y,z) = (0,0,0)$

so we take the ball $B = \{(x,y,z)|x^2+y^2+z^2 < 1\}$ in D and

consider the region $\Omega = D - B$ with $\partial\Omega = \partial D + \partial B$

Since V is differentiable in Ω, we can apply divergence theorem. (4 points)

$$\begin{aligned} divV &= \frac{\partial}{\partial x}(\frac{x}{r^3} + 2y + 3z) + \frac{\partial}{\partial y}(\frac{y}{r^3} + 3z)\frac{\partial}{\partial z}(\frac{z}{r^3} + x + 2y) \\ &= 0 \quad \text{(3 points)} \end{aligned}$$

$$\begin{aligned} &\Rightarrow \quad \int\int_{\partial\Omega} V \cdot \vec{n} d\sigma = \int\int\int_{\Omega} divV dV = 0 \\ &\Rightarrow \quad \int\int_{\partial D} V \cdot \vec{n} d\sigma = -\int\int_{\partial B} V \cdot \vec{n} d\sigma \quad \text{(2 points)} \end{aligned}$$

$\vec{n} = (-x,-y,-z)$ at ∂B , $-V \cdot \vec{n} = 1 + 3xy + 4xz + 5yz$ (1 point)

Let $\partial B^+ = \{(x,y,z) \in \partial B | z > 0\}$ and $\partial B^+ = \{(x,y,z) \in \partial B | z < 0\}$ $C = \{(x,y)|x^2+y^2 \le 1\}$

Then $-\int\int_{\partial B} V \cdot \vec{n} d\sigma = -\int\int_{\partial B+} V \cdot \vec{n} d\sigma + -\int\int_{\partial B-} V \cdot \vec{n} d\sigma$ (2 points)

Let $r^+(x,y) = (x,y,\sqrt{1-x^2-y^2})$ and $r^-(x,y) = (x,y,-\sqrt{1-x^2-y^2})$, Then $|r_x^+ \times r_y^+| = |r_x^- \times r_y^-| = \frac{1}{\sqrt{1-x^2-y^2}}$

$$\begin{aligned} -\int\int_{\partial B} V \cdot \vec{n} d\sigma &= -\int\int_{\partial B^+} V \cdot \vec{n} d\sigma + -\int\int_{\partial B^-} V \cdot \vec{n} d\sigma = \int\int_C 2\frac{1+3xy}{\sqrt{1-x^2-y^2}} dA \\ &= \int_0^{2\pi}\int_0^1 2\frac{1+3r^2\cos\theta\sin\theta}{\sqrt{1-r^2}} r dr d\theta \\ &= 4\pi \quad \text{(3 points)} \end{aligned}$$

7. (15%) Find $\mathbf{T}$ (unit tangent vector), $\mathbf{N}$ (principal unit vector), $\mathbf{B}$ (unit binormal vector), κ (curvature) and τ (torsion) for the space curve $\mathbf{r}(t) = (e^t \sin 2t)\mathbf{i} + (e^t \cos 2t)\mathbf{j} + 2e^t\mathbf{k}$ at $t = \dfrac{\pi}{2}$.

We calculate $\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)$ directly:

$$
\begin{aligned}
\mathbf{r}'(t) &= (e^t \sin 2t + 2e^t \cos 2t, -2e^t \sin 2t + e^t \cos 2t, 2e^t), & \mathbf{r}'(\tfrac{\pi}{2}) = e^{\frac{\pi}{2}}(-2,-1,2) \\
\mathbf{r}''(t) &= (-3e^t \sin 2t + 2e^t \cos 2t, -4e^t \sin 2t - 3e^t \cos 2t, 2e^t), & \mathbf{r}''(\tfrac{\pi}{2}) = e^{\frac{\pi}{2}}(-4,3,2) \\
\mathbf{r}'''(t) &= (-11e^t \sin 2t - 2e^t \cos 2t, 2e^t \sin 2t - 11e^t \cos 2t, 2e^t), & \mathbf{r}'''(\tfrac{\pi}{2}) = e^{\frac{\pi}{2}}(2,11,2)
\end{aligned}
$$

and $|\mathbf{r}'(t)| = 3e^t$

So, we have

$$
\begin{aligned}
\mathbf{T}|_{t=\frac{\pi}{2}} = \left.\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right|_{t=\frac{\pi}{2}} &= \left.\frac{(e^t \sin 2t + 2e^t \cos 2t, -2e^t \sin 2t + e^t \cos 2t, 2e^t)}{3e^t}\right|_{t=\frac{\pi}{2}} \\
&= \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)
\end{aligned}
$$

Since $\dfrac{d\mathbf{T}}{dt} = \left(\dfrac{2\cos 2t - 4\sin 2t}{3}, \dfrac{-4\cos 2t - 2\sin 2t}{3}, 0\right)$ we get the unit normal vector

$$
\mathbf{N}|_{t=\frac{\pi}{2}} = \left.\frac{\frac{d\mathbf{T}}{dt}}{|\frac{d\mathbf{T}}{dt}|}\right|_{t=\frac{\pi}{2}} = \frac{(-\frac{2}{3}, \frac{4}{3}, 0)}{\sqrt{(-\frac{2}{3})^2 + (\frac{4}{3})^2 + 0^2}} = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)
$$

By definition, we have $\mathbf{B}|_{t=\frac{\pi}{2}} = (\mathbf{T} \times \mathbf{N})|_{t=\frac{\pi}{2}} = \left(-4/2\sqrt{5}, -2/3\sqrt{5}, -5/3\sqrt{5}\right)$

Next, at $t = \frac{\pi}{2}$

$$
\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\right|\frac{1}{|\mathbf{r}'(t)|} = \frac{2\sqrt{5}}{9e^{\frac{\pi}{2}}}
$$

Finally, at $t = \frac{\pi}{2}$

$$
\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \wedge \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\begin{vmatrix} -2 & -1 & 2 \\ -4 & 3 & 2 \\ 2 & 11 & 2 \end{vmatrix}}{180e^{\frac{\pi}{2}}} = -\frac{4}{9e^{\frac{\pi}{2}}}
$$