1. (10 %) Find the area of the region that $1 \le r \le 3 + 3\cos\theta$.

   Sol.

$$\begin{cases} 3 + 3\cos\theta \ge 1 \\ 3\cos\theta \ge -2 \\ \cos\theta \ge \frac{-2}{3} \end{cases}$$

$\therefore \quad \theta$ range from $-\cos^{-1}(\frac{-2}{3})$ to $\cos^{-1}(\frac{-2}{3})$, Let $\theta_1 = \cos^{-1}(\dfrac{-2}{3})$

$\therefore \quad area = \displaystyle\int_{-\theta_1}^{\theta_1} \frac{(3+3\cos\theta)^2 - 1}{2} d\theta$

$= \displaystyle\int_0^{\theta_1} (3+3\cos\theta)^2 - 1 d\theta$

$= \dfrac{25}{2}\theta + 18\sin\theta + \dfrac{9}{4}\sin 2\theta \Big|_0^{\theta_1}$

$= \dfrac{25}{2}\theta_1 + 18\sin\theta_1 + \dfrac{9}{4}\sin(2\theta_1)$

$= 5\sqrt{5} + \dfrac{25}{2}\cos^{-1}(\dfrac{-2}{3})$

here we need to use

$\sin\theta_1 = \dfrac{\sqrt{5}}{3}$

$\sin(2\theta_1) = 2\sin\theta_1\cos\theta_1 = -\dfrac{4}{9}\sqrt{5}$

2. (10 %) Assume $x > 0$. Solve the initial value problem $x^2y' + 3y = \ln x$ with $y(1) = 0$.

   Sol.

$y' + \frac{3}{x}y = \frac{lnx}{x^2}$

then $v(x) = e^{\int \frac{3}{x}} = e^{3\ln x} = x^3$

so

$x^3\dot{y}' + x^3\dot{\frac{3}{x}}y = x\ln x$

we have $(y\dot{x}^3)' = x\ln x dx$

$(y\dot{x}^3) = \int x\ln x dx$

using integration by parts, so $x^3y = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x dx = \frac{1}{2}x^2\ln x - \frac{x^2}{4} + C$

and then, $y = \frac{1}{x^3}(\frac{1}{2}x^2\ln x - \frac{x^2}{4} + C) = \frac{\ln x}{2x} - \frac{1}{4x} + \frac{C}{x^3}$

with initial value $y(1) = 0$

so $C = \frac{1}{4}$

hence $y = \frac{1}{x^3}(\frac{1}{2}x^2 \ln x - \frac{x^2}{4} + C) = \frac{\ln x}{2x} - \frac{1}{4x} + \frac{1}{4x^3}$

3. (20 %) Given the ordinary differential equation $y' = y^2(y^2 - 1)$.

   (a) Identify the equilibrium values and state those which are stable and which are unstable.

   (b) Make a phase line of the differential equation and identify the signs of y' and y'' on it.

   (c) Sketch several solution curves.

   (d) Solve the equation by separation of variables. An equality in y and t is enough.

Sol.

(a) $y^2(y^2 - 1) = 0 \Leftrightarrow y = -1, 0, 1$

$y' < 0$ on $(-1, 1)$ and $y' > 0$ on $(-\infty, -1)$ and $(1, \infty)$

Hence -1 is stable, and 0,1 are unstable.

(b)

$$y'' = -2yy' + 4y^3y' = (4y^3 - 2y)y' = (4y^3 - 2y)y^2(y^2 - 1) = 2y^3(2y^2 - 1)(y^2 - 1)$$
$$= 4y^3(y - \frac{1}{\sqrt{2}})(y + \frac{1}{\sqrt{2}})(y - 1)(y + 1)$$

Thus, on $(-\infty, -1)$: $y' > 0, y'' < 0$

on $(-1, -\frac{1}{\sqrt{2}})$: $y' < 0, y'' > 0$

on $(-\frac{1}{\sqrt{2}}, 0)$: $y' < 0, y'' < 0$

on $(0, \frac{1}{\sqrt{2}})$: $y' < 0, y'' > 0$

on $(\frac{1}{\sqrt{2}}, 1)$: $y' < 0, y'' < 0$

on $(1, \infty)$: $y' > 0, y'' > 0$

(d) $y' = y^2(y^2 - 1)$

$\Rightarrow \dfrac{1}{y^2(y^2 - 1)}dy = dt$

$$\Rightarrow \left(\frac{-1}{y^2} + \frac{1}{2(y-1)} + \frac{-1}{2(y+1)}\right)dy = dt$$
$$\Rightarrow \int \frac{-1}{y^2}dy + \frac{1}{2}\int \frac{1}{y-1}dy - \frac{1}{2}\int \frac{1}{y+1}dy = \int dt$$
$$\Rightarrow \frac{1}{y} + \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| = t + \mathbf{C}$$

4. (15 %) At the point $P_0(1,3)$, a function $f(x,y)$ has a derivative of $-\sqrt{5}$ in the direction from $P_0(1,3)$ toward $P(2,1)$ and a derivative of $\sqrt{5}$ in the direction from $P_0(1,3)$ toward $P(3,2)$.

(a) Find the directions in which the function f increases and decreases most rapidly at $P_0(1,3)$. Then find the rates of change in these directions. (12%)

(b) Estimate how much the value of $f(x,y)$ will change if the point $P(x,y)$ moves from $P_0(1,3)$ straightly toward $P(0,0)$ with 0.05 unit. (3%)

Sol.

(a) Let $(\nabla f(x,y))_{P_0} = \nabla f(1,3) = f_x(1,3)\mathbf{i} + f_y(1,3)\mathbf{j}$. By the theorem of the directional derivative, we have the formula:

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}$$

Since the unit vector from $P_0(1,3)$ toward $P(2,1)$ is $\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$ and the unit vector from $P_0(1,3)$ toward $P(3,2)$ is $\frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{j}$, we get

$$\begin{cases} \frac{1}{\sqrt{5}}f_x(1,3) - \frac{2}{\sqrt{5}}f_y(1,3) = -\sqrt{5} \\ \frac{2}{\sqrt{5}}f_x(1,3) - \frac{1}{\sqrt{5}}f_y(1,3) = \sqrt{5} \end{cases} \Rightarrow \begin{cases} f_x(1,3) = 5 \\ f_y(1,3) = 5 \end{cases}$$

Thus, the direction in which the function f increases most rapidly at $P_0(1,3)$ is

$$\frac{\nabla f(1,3)}{|\nabla f(1,3)|} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j},$$

and the rate of change in this direction is $|\nabla f|_{P_0} = \sqrt{5^2+5^2} = 5\sqrt{2}$.

The direction in which the function f decreases most rapidly at $P_0(1,3)$ is

$$-\frac{\nabla f(1,3)}{|\nabla f(1,3)|} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j},$$

and the rate of change in this direction is $-|\nabla f|_{P_0} = -\sqrt{5^2+5^2} = -5\sqrt{2}$.

(b) Since the unit vector from $P_0(1,3)$ straightly toward $P(0,0)$ is $\mathbf{u} = -\frac{1}{\sqrt{10}}\mathbf{i} - \frac{3}{\sqrt{10}}\mathbf{j}$, we get

$$df = (\nabla f|_{P_0} \cdot \mathbf{u})(ds) = (5\mathbf{i} + 5\mathbf{j}) \cdot \left(-\frac{1}{\sqrt{10}}\mathbf{i} - \frac{3}{\sqrt{10}}\mathbf{j}\right)(0.05) = -\frac{1}{\sqrt{10}}.$$

5. (15 %) Let $w = f(x,y)$, $x = r\cos\theta$, and $y = r\sin\theta$. Express $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial^2 w}{\partial r^2}$ in terms of r, θ, and partial derivatives of $f(x,y)$ with respect to x and y.

Sol.

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = f_x \cos\theta + f_y \sin\theta$$

$$\frac{\partial^2 w}{\partial^2 r} = \frac{\partial}{\partial r}(f_x \cos\theta + f_y \sin\theta) = (\frac{\partial}{\partial r}(f_x)\cos\theta + \frac{\partial}{\partial r}(f_y)\sin\theta)$$

$$= (f_{xx}\sin\theta + f_{xy}\cos\theta)\sin\theta + (f_{yx}\cos\theta + f_{yy}\sin\theta)\sin\theta$$

$$= f_{xx}\cos^2\theta + (f_{xy} + f_{yx})\sin\theta\cos\theta + f_{yy}\cos^2\theta$$

$$(\text{or } = f_{xx}\cos^2\theta + 2f_{xy}\sin\theta\cos\theta + f_{yy}\cos^2\theta)$$

6. (10 %) Assume $f(x,y) = 3xy^2 - x^3 - \frac{6}{5}y^5$. Find all the local maximum, local minimum and saddle points of f. Please still determine the types of critical points even if the second derivative test is inconclusive.

Sol.

$$\begin{cases} f_x = 3y^2 - 3x^2 = 0 \\ f_y = 6xy - 6y^4 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = \pm y \\ x = y^3 \text{ or } y = 0 \end{cases} \Rightarrow (x,y) = (0,0), (-1,-1), (1,1)$$

Second derivative test:

$f_{xx} = -6x$, $f_{yy} = 6x - 24y^3$, $f_{xy} = 6y = f_{yx}$

$$H_f = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2 = -36(x^2 - 4xy^3 + y^2)$$

$(0,0) : H_f(0,0) = 0 \Rightarrow$ inconclusive,

consider $f(x,0) = -x^3 \Rightarrow f(x,0) > 0$ when $x < 0, f(x,0) < 0$ when $x > 0$

$\Rightarrow (0,0)$ is a saddle point.

$(-1,-1) : H_f(-1,-1) = 72 > 0$ and $f_{xx}(-1,-1) = 6 > 0$

$\Rightarrow (-1,-1)$ is a local minimum point.

$(1,1) : H_f(1,1) = 72 > 0$ and $f_{xx}(1,1) = -6 < 0$

$\Rightarrow (1,1)$ is a local maximum point.

7. (20 %) Suppose Γ is the intersection curve of $x + y + 2z = 2$ and $z^2 = 2x^2 + 2y^2$.

   (a) Find the tangent line of Γ at $(\frac{1}{2}, \frac{-1}{2}, 1)$.

   (b) Find all the points on Γ that lie closest and farthest from the origin.

Sol.

(a) Let $f(x,y,z) = x + y + 2z - 2, g(x,y,z) = 2x^2 + 2y^2 - z^2, P = (\frac{1}{2}, -\frac{1}{2}, 1)$

$\nabla f = (1,1,2), \nabla f|_P = (1,1,2)$

$\nabla g = (4x, 4y, -2z), \nabla g|_P = (2,-2,-2)$

The direction of tangent vector of Γ is

$\nabla f|_P \times \nabla g|_P = (1,1,2) \times (2,-2,-2) = (2,6,-4)$

The tangent line of Γ at $(\frac{1}{2}, \frac{-1}{2}, 1)$ is

$$\begin{cases} x = \frac{1}{2} + 2t \\ y = -\frac{1}{2} + 6t \quad t \in \mathbb{R} \\ z = 1 - 4t \end{cases}$$

(b)

$d(x,y,z) = x^2 + y^2 + z^2, \nabla d = \lambda \nabla f + s \nabla g$ and $f = 0, g = 0 \Rightarrow$

$$\begin{cases} 2x = \lambda + 4sx \\ 2y = \lambda + 4sy \\ 2z = 2\lambda - 2sz \\ f = 0 \\ g = 0 \end{cases} \Rightarrow \begin{cases} (2-4s)x = \lambda \\ (2-4s)y = \lambda \\ (2+2s)z = \lambda \\ 2x^2 + 2y^2 - z^2 = 0 \\ x + y + 2z - 2 = 0 \end{cases}$$

Case1. $(2-4s) = 0 \Rightarrow \lambda = 0 \Rightarrow z = 0 \Rightarrow \begin{cases} 2x^2 + 2y^2 = 0 \\ x + y = 2 \end{cases} \Rightarrow x = y = 0$ and $x + y = 2$, no solution

Case2. $(2-4s) \neq 0 \Rightarrow x = y \Rightarrow \begin{cases} 4x^2 = z^2 \\ 2x + 2z = 2 \end{cases} \Rightarrow z = \pm 2x$ and $2x + 2z = 2$

If $z = 2x \Rightarrow (x,y,z) = (\frac{1}{3}, \frac{1}{3}, \frac{2}{3})$

If $z = -2x \Rightarrow (x,y,z) = (-1,-1,2)$

At $(x,y,z) = (\frac{1}{3}, \frac{1}{3}, \frac{2}{3}) \Rightarrow d = \frac{2}{3}$

At $(x,y,z) = (-1,-1,2) \Rightarrow d = 6$

So $(\frac{1}{3}, \frac{1}{3}, \frac{2}{3})$ is the closest point from $(0,0,0)$

So $(-1,-1,2)$ is the farthest point from $(0,0,0)$