1. (10%)

求極限: $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$ (提示: 取對數)

解:

$$\ln \frac{\sqrt[n]{n!}}{n} = \frac{1}{n}\ln\left(\frac{n!}{n^n}\right) = \frac{1}{n}\{\ln\frac{1}{n} + \ln\frac{2}{n} + ... + \ln\frac{n}{n}\}$$

Hence

$$\lim_{n->\infty} \ln \frac{\sqrt[n]{n!}}{n} = \int_0^1 (\ln x)dx = \lim_{\delta->0^+} \int_\delta^1 (\ln x)dx = \lim_{\delta->0^+} x\ln x|_\delta^1 - \int_\delta^1 x\cdot\frac{dx}{x}$$

$$= \lim_{\delta->0^+} x\ln x|_\delta^1 - x|_\delta^1 = \lim_{\delta->0^+} -\delta\ln\delta - (1-\delta) = -1 + \lim_{\delta->0^+} \delta - \delta\ln\delta = -1 - \lim_{t->\infty} \frac{-\ln t}{t} = -1$$

where $t = \frac{1}{\delta}$. And since e^x is continuous

$$\lim_{n->\infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n->\infty} e^{\ln \frac{\sqrt[n]{n!}}{n}} = e^{\lim_{n->\infty} \ln \frac{\sqrt[n]{n!}}{n}} = e^{-1}$$

2. (10%)

求極限: $\lim\limits_{x\to\infty} \dfrac{\int_0^{x^2} e^{t-x^2}(2t^2+1)dt}{x^4}$

解:

$$
\begin{aligned}
&\lim_{x\to\infty} \frac{\int_0^{x^2} e^{t-x^2}(2t^2+1)dt}{x^4} \\
=&\lim_{x\to\infty} \frac{(2t^2+1)e^{t-x^2}|_0^{x^2} - \int_0^{x^2} 4te^{t-x^2}dt}{x^4} \\
=&\lim_{x\to\infty} \frac{(2t^2+1)e^{t-x^2}|_0^{x^2} - (4te^{t-x^2}|_0^{x^2} - 4\int_0^{x^2} e^{t-x^2}dt)}{x^4} \\
=&\lim_{x\to\infty} \frac{(2t^2-4t+5)e^{t-x^2}|_0^{x^2}}{x^4} \\
=&\lim_{x\to\infty} \frac{2x^4-4x^2+5-5e^{-x^2}}{x^4} \\
=&2.
\end{aligned}
$$

3. (10%)
求心臟線 $r = 1 + \cos\theta$ 在 x 軸上方之最高高度。

解:
令$y = (1+\cos\theta)\sin\theta$.

$$\begin{aligned} 0 &= \frac{dy}{d\theta} \\ &= \cos^2\theta + \cos\theta - \sin^2\theta \\ &= \cos^2\theta + \cos\theta - (1-\cos^2\theta) \\ &= 2\cos^2\theta + \cos\theta - 1 \end{aligned}$$

所以 $\cos\theta = \frac{1}{2}, -1$(不合), 得到 $\theta = \frac{\pi}{3}$。因此最高點之高度爲 $(1+\cos\theta)\sin\theta = \frac{3\sqrt{3}}{4}$.

4. (20%)

設 Ω 爲曲線 $y=\sin x$ 及 x 軸, 在區間 $0 \le x \le \frac{\pi}{2}$ 上所圍的區域。

a. 求 Ω 繞 x 軸旋轉之體積。

b. 求 Ω 繞 y 軸旋轉之體積。

c. 求 Ω 之形心。

d. 求 Ω 繞著直線 $8x+6y=25$ 旋轉之體積。

(a)

$$\int_0^{\frac{\pi}{2}} \pi y^2 dx = \int_0^{\frac{\pi}{2}} \pi \sin x^2 dx = \int_0^{\frac{\pi}{2}} \pi (\frac{1-\cos 2x}{2}) dx = \pi (\frac{x}{2} - \frac{\sin x}{4}) \Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{4}$$

(b)

Method1: (剝殼)

$$2\pi \int_0^{\frac{\pi}{2}} x \sin x dx = 2\pi (-x\cos x + \sin x) \Big|_0^{\frac{\pi}{2}} = 2\pi$$

Method2: (反扣)

$$\pi (\frac{\pi}{2})^2 - \pi \int_0^1 x^2 dy = 2\pi$$

(c)

Method 1 : 利用轉動慣量

$$Area = \int_0^{\frac{\pi}{2}} \sin x dx = 1$$

$$M_x = \int_0^{\frac{\pi}{2}} x \sin x dx = 1$$

$$M_y = \int_0^{\frac{\pi}{2}} \frac{y^2}{2} dx = \frac{\pi}{8}$$

Method2: 利用 Pappus theorem

$$2\pi M_x A = \overline{x} \times 繞Y軸體積$$

$$2\pi M_y A = \overline{y} \times 繞X軸體積$$

(d)

Pappus theorem

$$\left| \frac{8 \times 1 + 6 \times \frac{\pi}{8} - 25}{\sqrt{6^2 + 8^2}} \right| = \frac{17 - 3\pi}{10}$$

$$2\pi \times \frac{17-3\pi}{10} \times 1 = \frac{17}{5}\pi - \frac{3}{20}\pi^2$$

5. (10%)

求積分:$\displaystyle\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}}dx$

解:

因爲

$$\frac{d}{dx}\sin^{-1}\sqrt{x} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

所以

$$\begin{aligned}
\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}}dx &= \int_{\frac{1}{4}}^{\frac{1}{2}} 2\sin^{-1}\sqrt{x} \quad d\sin^{-1}\sqrt{x} \\
&= (\sin^{-1}\sqrt{x})^2]_{\frac{1}{4}}^{\frac{1}{2}} \\
&= \left(\frac{\pi}{4}\right)^2 - \left(\frac{\pi}{6}\right)^2 \\
&= \frac{5}{144}\pi^2.
\end{aligned}$$

6. (10%)

求積分:$\displaystyle\int \frac{1}{2+\sin x}dx$

解:

Let $u = \tan(\frac{x}{2})$, then $\sin x = \frac{2u}{1+u^2}$, $dx = \frac{2}{1+u^2}du$

$\int \frac{1}{2+\sin x} \ = \ \int \frac{1}{2+\frac{2u}{1+u^2}} \frac{2}{1+u^2}du \ = \ \int \frac{du}{u^2+u+1} \ = \ \int \frac{du}{(u+\frac{1}{2})^2+\frac{3}{4}} \ = \ \frac{4}{3}\int \frac{du}{(\frac{2u+1}{\sqrt{3}})^2+1}$

Let $\frac{2u+1}{\sqrt{3}} = \tan\theta$, then $du = \frac{\sqrt{3}}{2}\sec^2\theta d\theta$

$\int \frac{1}{2+\sin x} \ = \ \frac{4}{3}\int \frac{du}{(\frac{2u+1}{\sqrt{3}})^2+1} \ = \ \frac{4}{3}\int \frac{\frac{\sqrt{3}}{2}\sec^2\theta d\theta}{\tan^2\theta+1} \ = \ \frac{2}{\sqrt{3}}\int 1 d\theta$

$= \ \frac{2}{\sqrt{3}}(\theta + C_1) \ = \ \frac{2}{\sqrt{3}}(\tan^{-1}(\frac{2u+1}{\sqrt{3}}) + C_1) \ = \ \frac{2}{\sqrt{3}}\tan^{-1}(\frac{2\tan(\frac{x}{2})+1}{\sqrt{3}}) + C$

7. (10%)

求積分: $\displaystyle\int \frac{dx}{x^3\sqrt{x^2+3}}$

解:

$\int \frac{dx}{x^3\sqrt{x^2+3}} = \int \frac{x}{x^4\sqrt{x^2+3}}dx = \int \frac{1}{(u^2-3)^2}du$

Since $\frac{1}{(u^2-3)^2} = \frac{1}{12\sqrt{3}(u+\sqrt{3})} - \frac{1}{12\sqrt{3}(u-3)} + \frac{1}{12(u+\sqrt{3})^2} + \frac{1}{12(u-\sqrt{3})^2}$,

$\int \frac{dx}{x^3\sqrt{x^2+3}}$

$= \int \frac{1}{12\sqrt{3}(u+\sqrt{3})}du - \int \frac{1}{12\sqrt{3}(u-3)}du + \int \frac{1}{12(u+\sqrt{3})^2}du + \int \frac{1}{12(u-\sqrt{3})^2}du$

$= \frac{1}{12\sqrt{3}}(\ln(u+\sqrt{3}) - \ln(u-\sqrt{3})) - \frac{1}{12}(\frac{1}{u+\sqrt{3}} + \frac{1}{u-\sqrt{3}})$

$= \frac{1}{12\sqrt{3}}\ln(\frac{\sqrt{x^2+3}+\sqrt{3}}{\sqrt{x^2+3}-\sqrt{3}}) - \frac{1}{6}\frac{\sqrt{x^2+3}}{x^2}$

8. (10%)

設瑕積分 $\displaystyle\int_{\sqrt{2}}^{\infty}(\frac{a}{\sqrt{x^2-1}}-\frac{x}{x^2+1})dx$ 爲收斂。試決定a值, 並求其積分值。

解:

$\int_{\sqrt{2}}^{\infty}(\frac{a}{\sqrt{x^2-1}}-\frac{x}{x^2+1})dx$

$=\lim_{b\to\infty}\int_{\sqrt{2}}^{b}(\frac{a}{\sqrt{x^2-1}}-\frac{x}{x^2+1})dx$

$=\lim_{b\to\infty}(a\ln|x+\sqrt{x^2-1}|-\frac{1}{2}\ln|x^2+1|)|_{\sqrt{2}}^{b}$

$=\lim_{b\to\infty}\ln\frac{(|b+\sqrt{b^2-1}|)^a}{\sqrt{b^2+1}}-\ln|1+\sqrt{2}|+\frac{1}{2}\ln 3$

and

$\lim_{b\to\infty}\ln\frac{(|b+\sqrt{b^2-1}|)^a}{\sqrt{b^2+1}}$

need to be convergent.

If $a>1$,it is divergent as $b\to\infty$;

If $a<1$,it is divergent since $\ln 0\to-\infty$as $b\to\infty$;

The value of a must be 1.

and

$\lim_{b\to\infty}\ln\frac{(|b+\sqrt{b^2-1}|)}{\sqrt{b^2+1}}=\ln 2$

The value of the improper integral is

$\ln 2-\ln|1+\sqrt{2}|+\frac{1}{2}\ln 3=\ln(\frac{2\sqrt{3}}{1+\sqrt{2}})$

9. (10%)

解微分方程 $xy' - 2y = x^3 \sec x \tan x,\ x > 0,\ y(\dfrac{\pi}{4}) = 0$ 。

解:
$xy' - 2y = x^3 \sec x \tan x;$
同除x^3,$(y/x^2)' = y'/x^2 - 2y/x^3 = \sec x \tan x;$
故$y/x^2 = \sec x + C;$
因$y(\frac{\pi}{4}) = 0$, 故$C = -\sqrt{2}$。
得$y = x^2(\sec x - \sqrt{2})$
評分標準:
$y' - 2y/x = x^2 \sec x \tan x$: 至此一分;
用積分因子法, 令$(vy)' = vy' - 2vy/x = vx^2 \sec x \tan x$, 並解出$v = 1/x^2$: 至此五分;
求得 $y/x^2 = \sec x + C$: 至此八分;
得到 $y = x^2(\sec x + C)$; 而 C 解錯, 至此九分。
若用參數變異法, 則求出齊次解等同解出積分因子, 給五分。
積分因子法推導中間有誤, 酌量扣分, 但直接用錯誤的公式不給分。