1. (18%) Evaluate the limits

(a) $\lim\limits_{x\to 0^+}(e^x-1)^{\frac{1}{\ln x}}$

sol.

Let $f(x)=(e^x-1)^{\frac{1}{\ln x}} \Rightarrow \ln f(x)=\frac{\ln(e^x-1)}{\ln x}$ $\lim\limits_{x\to 0^+}\ln f(x)=\lim\limits_{x\to 0^+}\frac{\ln(e^x-1)}{\ln x}=$ $\lim\limits_{x\to 0^+}\frac{e^x/(e^x-1)}{1/x}=\lim\limits_{x\to 0^+}\frac{xe^x}{e^x-1}=\lim\limits_{x\to 0^+}\frac{e^x+xe^x}{e^x}=\lim\limits_{x\to 0^+}\frac{1+x}{1}=1$

Because e^z is continuous at z, $\lim\limits_{x\to 0^+}(e^x-1)^{\frac{1}{\ln x}}=\lim\limits_{x\to 0^+}f(x)=\lim\limits_{x\to 0^+}e^{\ln f(x)}=$ $e^{\lim\limits_{x\to 0^+}\ln f(x)}=e^1$

(b) $\lim\limits_{x\to 0}\frac{1}{x^4}\int_0^{x^2}\frac{t}{t^2+1}dt$

sol.

$$\lim_{x\to 0}\frac{1}{x^4}\int_0^{x^2}\frac{t}{t^2+1}dt=\lim_{x\to 0}\frac{\frac{x^2}{x^4+1}\cdot 2x}{4x^3}=\lim_{x\to 0}\frac{1}{2(x^4+1)}=\frac{1}{2}$$

(c) $\lim\limits_{n\to\infty}(\frac{n}{n^2+4}+\frac{n}{n^2+16}+\frac{n}{n^2+36}+\cdots+\frac{n}{n^2+4k^2}+\cdots+\frac{n}{5n^2})$

sol.

$$\lim_{n\to\infty}(\frac{n}{n^2+4}+\frac{n}{n^2+16}+\frac{n}{n^2+36}+\cdots+\frac{n}{n^2+4k^2}+\cdots+\frac{n}{5n^2})=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{1+(2k/n)^2}\cdot\frac{1}{n}=\int_0^1\frac{dx}{1+(2x)^2}=\frac{1}{2}\tan^{-1}2x|_0^1=\frac{1}{2}\tan^{-1}2$$

2. (18%)

(a) Find the equations of the tangent line and the normal line to the curve $3x^2 - \frac{4y}{x} = 5$ at $x = 3$.

sol.

$x = 3 \Rightarrow 27 - \frac{4y}{3} = 5 \Rightarrow y = \frac{33}{2}$

$3x^2 - \frac{4y}{x} = 5 \Rightarrow 6x - \frac{4xy' - 4y}{x^2} = 0 \Rightarrow 18 - \frac{12y' - 66}{9} = 0 \Rightarrow y' = 19$

tangent line: $y - \frac{33}{2} = 19(x-3)$; normal line: $y - \frac{33}{2} = \frac{-1}{19}(x-3)$

(b) Let $y = x^{a^b} + a^{x^b} + a^{b^x}$, $a > 0$, $b > 0$. Find $\frac{dy}{dx}$.

sol.

$\frac{d}{dx}(x^{a^b}) = a^b x^{a^b - 1}$

Let $y = a^{x^b} \Rightarrow \ln y = x^b \ln a \Rightarrow \frac{1}{y}y' = bx^{b-1}\ln a \Rightarrow y' = b(\ln a)x^{b-1}a^{x^b}$

Let $y = a^{b^x} \Rightarrow \ln y = b^x \ln a \Rightarrow \ln(\ln y) = x\ln b + \ln(\ln a)$

$\frac{1}{\ln y}\frac{1}{y}y' = \ln b \Rightarrow y' = (\ln b)(\ln y)y = (\ln b)(\ln a)b^x a^{b^x}$

$\therefore \frac{d}{dx}(x^{a^b} + a^{x^b} + a^{b^x}) = a^b x^{a^b-1} + b(\ln a)x^{b-1}a^{x^b} + (\ln b)(\ln a)b^x a^{b^x}$

(c) Let $f(x) = \frac{x\text{cos}x}{(x+1)(x+2)(x+3)\cdots(x+n)}$. Find $f'(0)$.

sol.

$f'(x) = \frac{\cos x}{(x+1)\cdots(x+n)} + x\frac{d}{dx}(\frac{\cos x}{(x+1)\cdots(x+n)})$

$\therefore f'(0) = \frac{1}{n!}$

3. (12%)

(a) Find the derivative of the function $f(x) = \int_{2x^2+1}^{x^3} te^t dt$.

sol. Use Fundamental Theorem of Calculus,

$\frac{d}{dx}f(x) = x^3 e^{x^3} 3x^2 - (2x^2+1)e^{2x^2+1}4x$

$= 3x^5 e^{x^3} - (8x^3+4x)e^{2x^2+1}$

(b) Find $\dfrac{dy}{dx}$ if $y + x = \displaystyle\int_y^{y^2} \sin(t^2)dt + \int_x^{x^2} \cos\sqrt{t}dt$.

sol. 等式兩邊同時對x微分,

$$\Rightarrow \frac{dy}{dx} + 1 = 2y\frac{dy}{dx}sin(y^4) - \frac{dy}{dx}sin(y^2) + 2xcos|x| - cos\sqrt{x}$$

$$\Rightarrow (sin(y^2) - 2ysin(y^4) + 1)\frac{dy}{dx} = 2xcos|x| - cos\sqrt{x} - 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xcos|x| - cos\sqrt{x} - 1}{sin(y^2) - 2ysin(y^4) + 1}$$

4. (12%)

(a) Give an $\varepsilon - \delta$ proof to show that $\lim_{x \to a} f(x) = L$ implies $\lim_{x \to a} |f(x)| = |L|$.

sol.

For any$\varepsilon > 0$,there is a $\delta > 0$ such that if $0 < |x - a| < \delta$

Then $|f(x) - L| < \varepsilon$

Choose the same δ,whenever $0 < |x - a| < \delta$

$||f(x)| - |L||^2 = (|f(x)| - |L|)^2 = f(x)^2 + L^2 - 2|f(x)L|$

$= (f(x) - L)^2 + 2(f(x)L - |f(x)L|)$

$\leq |f(x) - L|^2 + 2 \cdot 0 < \varepsilon^2$

Then$||f(x)| - |L|| < \varepsilon$

That is

$$\lim_{x \to a} |f(x)| = |L|.$$

(b) Show that the reversed direction of the above statement is not true.

sol.

Let $f(x) = 1$for $x \geqq 0, f(x) = -1$for $x < 0$and $|f(x)| = 1$

Since

$$\lim_{x \to 0} |f(x)| = 1 = L$$

However $\lim_{x \to 0^+} f(x) = 1 = L$,

$\lim_{x \to 0^-} f(x) = -1 \neq L$

5. (8%)

Find the total area of the region between the curves $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq \frac{3}{2}\pi$.

sol.

Let A be the area of the region between $\sin x$ and $\cos x$ for $0 \leq x \leq \frac{3}{2}\pi$.

$$A = \int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)dx + \int_{5\pi/4}^{3\pi/2} (\cos x - \sin x)dx$$

$$= (\sin x + \cos x)|_0^{\pi/4} + (-\cos x - \sin x)|_{\pi/4}^{5\pi/4} + (\sin x + \cos x)|_{5\pi/4}^{3\pi/2}$$

$$= (\sqrt{2} - 1) + (\sqrt{2} + \sqrt{2}) + (-1 + \sqrt{2})$$

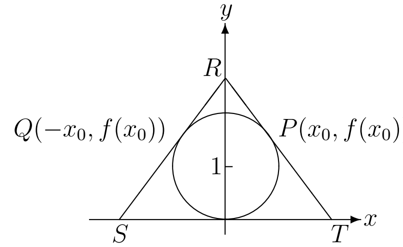
$$= 4\sqrt{2} - 2.$$

6. (12%)

Let $f(x)$ be the function representing the upper half of the circle $x^2+(y-1)^2=1$ centered at $(0,1)$. A triangle RST is formed by the tangent lines to the circle at $P(x_0, f(x_0))$ and $Q(-x_0, f(x_0))$ as shown in the figure, where $x_0 \in (0,1)$.

(a) Show that the area of the triangle is $A(x_0) = -f'(x_0)\left[x_0 - \frac{f(x_0)}{f'(x_0)}\right]^2$.

(b) Determine the height of the triangle with minimum area.



sol.

(a) Tangent line: $y = f(x_0) + f'(x_0)(x - x_0)$,

x-intercept: $x_i = x_0 - \frac{f(x_0)}{f'(x_0)}$,

y-intercept: $y_i = f(x_0) - x_0 f'(x_0)$.

The area of the triangle $\triangle RST$ is

$A(x_0) = x_i y_i = (x_0 - \frac{f(x_0)}{f'(x_0)})(f(x_0) - x_0 f'(x_0)) = -f'(x_0)(x_0 - \frac{f(x_0)}{f'(x_0)})^2$.

(b) Let O be the origin, and let $h = \overline{OR} > 2$.

Thus, $\overline{PR}^2 + 1^2 = (h-1)^2$ and $\overline{OT} : h = 1 : \overline{PR}$.

$A(h) = \overline{OT} \cdot \overline{OR} = \frac{h^2}{\sqrt{h^2-2h}}$, $A'(h) = \frac{h^2(h-3)}{(h^2-2h)^{3/2}}$.

$A'(h) = 0 \Rightarrow h = 3$.

Since $h^2 > 0$ and $(h^2-2h)^{3/2} > 0$, we have $A'(h) < 0$ when $h < 3$ and $A'(h) > 0$ when $h > 3$, that is, $A(h)$ decreases when $h < 3$, and increases when $h > 3$.

So $h = 3$ minimizes $A(h)$.

7. (20%)

Let $f(x) = \dfrac{x^3 - x - 1}{x^2}$. Answer the following questions.

(a) The domain of $y = f(x)$ is $\underline{\quad \mathrm{R}\setminus\{0\} \quad}$.

(b) $f'(x) = \underline{\quad 1 + \frac{1}{x^2} + \frac{2}{x^3} = \frac{x^3+x+2}{x^3} = \frac{(x+1)(x^2-x+2)}{x^3} \quad}$.

(c) $y = f(x)$ has critical point(s) at $\underline{\quad x = -1}$.

(d) $f''(x) = \underline{\quad -\frac{2}{x^3} - \frac{6}{x^4} = \frac{-2(x+3)}{x^4} \quad}$.

(e) $y = f(x)$ is increasing on interval(s) $\underline{\quad (-\infty, -1) \text{ and } (0, \infty) \quad}$,

$y = f(x)$ is decreasing on interval(s) $\underline{\quad (-1, 0) \quad}$.

(f) $y = f(x)$ is concave up on interval(s) $\underline{\quad (-\infty, -3) \quad}$,

$y = f(x)$ is concave down on interval(s) $\underline{\quad (-3, 0) \text{ and } (0, \infty) \quad}$.

(g) Find the (x, y)-coordinates of the following points if exist.

Local maximum point(s) : $\underline{\quad (-1, -1) \quad}$,

Local minimum point(s) : $\underline{\quad \text{none} \quad}$,

Inflection point(s) : $\underline{\quad (-3, \frac{-25}{9}) \quad}$.

(h) Find the asymptotes of the graph $y = f(x)$ if exist.

Vertical asymptotes(s) : $\underline{\quad x = 0 \quad}$,

Horizontal asymptotes(s) : $\underline{\quad \text{none} \quad}$ as $x \to$ $\underline{\quad \text{none} \quad}$,

Slanted asymtotes(s) : $\underline{\quad y = x \quad}$ as $x \to$ $\underline{\quad \infty \text{ or } -\infty \quad}$.

(i) Sketch the graph of $y = f(x)$ on the next page.

ans:

$$\text{The domain of } f(x) \text{ is } \mathrm{R}\backslash\{0\}$$

$$f(x) = \frac{x^3 - x - 1}{x^2} = x - \frac{1}{x} - \frac{1}{x^2}$$

$$f'(x) = 1 + \frac{1}{x^2} + \frac{2}{x^3} = \frac{x^3 + x + 2}{x^3} = \frac{(x+1)(x^2 - x + 2)}{x^3}$$

$$\because \qquad f'(x) = 0 \qquad \Rightarrow \qquad x = -1$$

$$\therefore \qquad f(x) \text{ has critical point at } x = -1$$

$$\text{and } f'(x) > 0 \text{ on } (-\infty, -1) \text{ and } (0, \infty)$$

$$f'(x) < 0 \text{ on } (-1, 0)$$

$$\therefore \qquad f(x) \text{ increasing on } (-\infty, -1) \text{ and } (0, \infty) \text{ and } \qquad f(x) \text{ decreasing on } (-1, 0)$$

$$f''(x) = -\frac{2}{x^3} - \frac{6}{x^4} = \frac{-2(x+3)}{x^4}$$

$$\because \qquad f''(x) = 0 \qquad \Rightarrow \qquad x = -3$$

$$\text{and } f''(x) > 0 \text{ on } (-\infty, -3)$$

$$f''(x) < 0 \text{ on } (-3, 0) \text{ and } (0, \infty)$$

$$\therefore \qquad f(x) \text{ concave up on } (-\infty, -3) \text{ and } \qquad f(x) \text{ concave down on } (-3, 0) \text{ and } (0, \infty)$$

$$\text{since } f'(-1) = 0, \text{ and } f''(-1) < 0,$$

$$\therefore \qquad f(x) \text{ has a local maximum point at} (x, y) = (-1, -1)$$

$$\text{and } f'(-3) > 0, \text{ and } f''(-3) = 0,$$

$$\therefore \qquad f(x) \text{ has inflection point at}(x,y) = (-3, \frac{-25}{9})$$

$$f(x) \text{ has no local minimum points}$$

$$\because \qquad f(x) = x - \frac{1}{x} - \frac{1}{x^2}$$

$$\therefore \qquad f(x) \text{ approaches to } y = x \text{ as } x \to \infty \text{ and } x \to \ -\infty$$

$$\therefore \qquad f(x) \text{ has a slanted asymptote } y = x$$

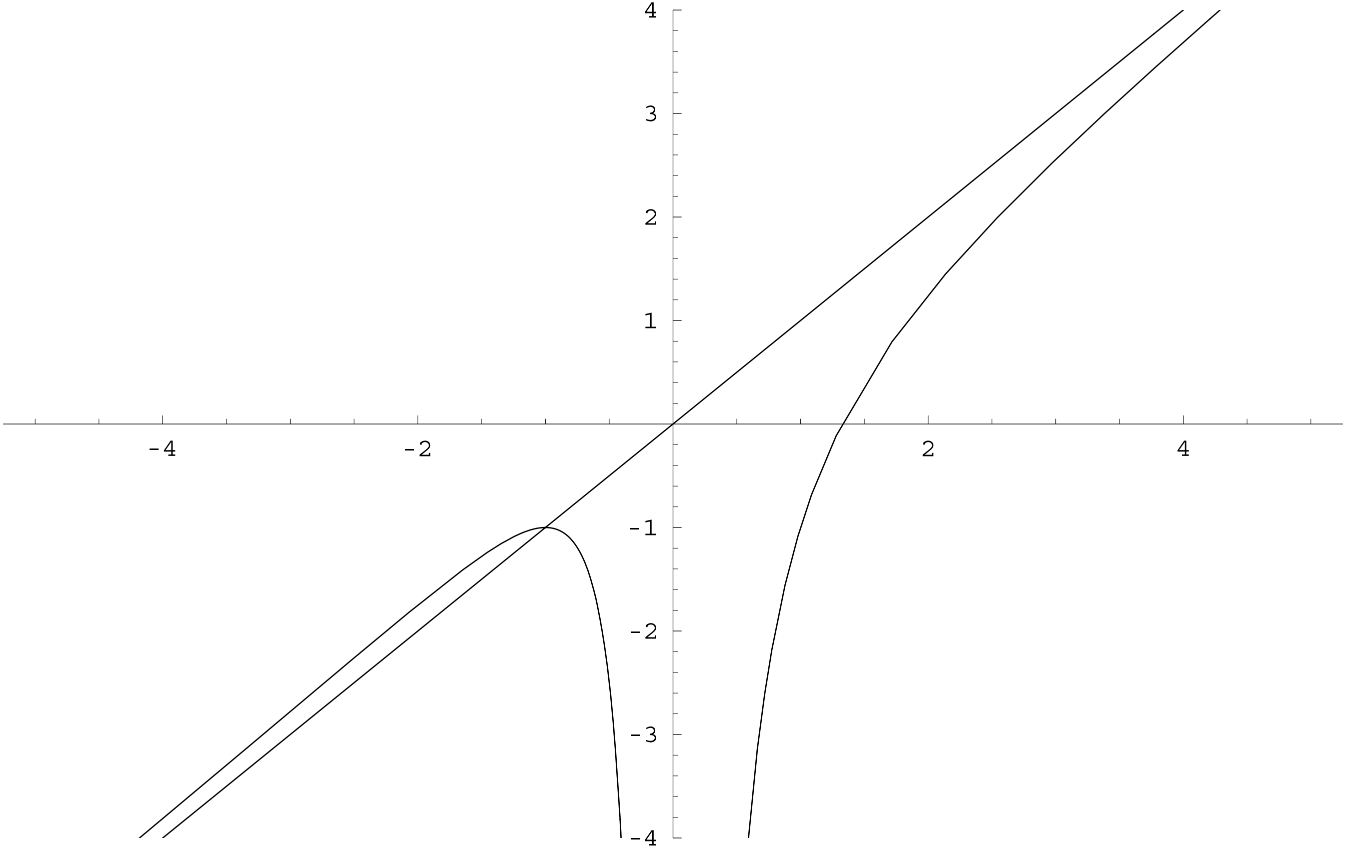
$$\text{and } \lim_{x \to 0} f(x) = -\infty$$

$$\therefore \qquad f(x) \text{ has a vertical asymptote } x = 0$$

$$f(x) \text{ has no horizontal asymptotes}$$

Figure 1: $f(x) = \frac{x^3-x-1}{x^2}$

圖形見下頁。



4
3
2
1
-1
-2
-3
-4
-4
-2
2
4