1. (24%) Evaluate the integral.

(a) $\int \sin^2 \sqrt{x} dx$ (b) $\int \frac{2\sqrt{x+2}}{(x+1)^2} dx$ (c) $\int \frac{x dx}{\sqrt{x^2+x+1}}$

sol. (a) Let $\sqrt{x} = u$, $dx = 2udu$,

$$\int \sin^2 \sqrt{x} dx = \int \sin^2 u \cdot 2udu$$
$$= \int \frac{1-\cos 2u}{2} \cdot 2udu$$
$$= \int udu - \int u \cos 2udu$$
$$= \frac{u^2}{2} - (\frac{u}{2} \cdot \sin 2u - \int \frac{1}{2} \sin 2udu)$$
$$= \frac{u^2}{2} - \frac{u}{2} \cdot \sin 2u - \frac{1}{4} \cos 2u + C$$
$$= \frac{x}{2} - \frac{\sqrt{x}}{2} \cdot \sin 2\sqrt{x} - \frac{1}{4} \cos 2\sqrt{x} + C$$

(b)Let $\sqrt{x+2} = u$, $dx = 2udu$,

$$\int \frac{2\sqrt{x+2}}{(x+1)^2} dx = \int \frac{2u}{(u^2-1)^2} \cdot 2udu = \int \frac{4u^2}{(u^2-1)^2} du$$
$$= \int (\frac{1}{u-1} + \frac{-1}{u+1} + \frac{1}{(u-1)^2} + \frac{1}{(u+1)^2}) du$$
$$= \ln|u-1| - \ln|u+1| - \frac{1}{u-1} - \frac{1}{u+1} + C$$
$$= \ln|\sqrt{x+2}-1| - \ln|\sqrt{x+2}+1| - \frac{1}{\sqrt{x+2}-1} - \frac{1}{\sqrt{x+2}+1} + C$$
$$= -2(\tanh^{-1}(\sqrt{x+2}) + \frac{\sqrt{x+2}}{x+1}) + C$$

(c) $$\int \frac{x dx}{\sqrt{x^2+x+1}} = \frac{1}{2} \int \frac{2x+1}{\sqrt{x^2+x+1}} dx - \frac{1}{2} \int \frac{dx}{\sqrt{x^2+x+1}}$$
$$= \frac{1}{2} \int \frac{d(x^2+x+1)}{\sqrt{x^2+x+1}} - \frac{1}{2} \int \frac{dx}{\sqrt{(\frac{\sqrt{3}}{2}) + (x+\frac{1}{2})^2}}$$
$$= \sqrt{x^2+x+1} - \frac{1}{2} \sinh^{-1}(\frac{2x+1}{\sqrt{3}}) + C$$

2. (16%) Determine whether the integral converges or diverges. Give reasons for your answers.

(a) $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}+x^3}$ (b) $\displaystyle\int_{-1}^{1} \frac{x+1}{\sqrt[5]{x^6}}dx$

sol.

(a) $\displaystyle 0 \le \int_0^{\infty} \frac{dx}{\sqrt{x}+x^3}$

$= \int_1^{\infty} \frac{dx}{\sqrt{x}+x^3} + \int_0^1 \frac{dx}{\sqrt{x}+x^3}$

$\le \int_1^{\infty} \frac{dx}{x^3} + \int_0^1 \frac{dx}{\sqrt{x}}$

$= -\frac{1}{2x^2}|_1^{\infty} + 2\sqrt{x}|_0^1$

$= \frac{1}{2} + 2$

$= \frac{5}{2} < \infty$

Then $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}+x^3}$ converges

(b) $\displaystyle\int_{-1}^{1} \frac{x+1}{\sqrt[5]{x^6}}dx$

$\displaystyle= \int_{-1}^{0} \frac{x+1}{\sqrt[5]{x^6}}dx + \int_{0}^{1} \frac{x+1}{\sqrt[5]{x^6}}dx$

$\displaystyle= \int_{-1}^{0} (x^{\frac{-1}{5}} + x^{\frac{-6}{5}})dx + \int_{0}^{1} (x^{\frac{-1}{5}} + x^{\frac{-6}{5}})dx$

$= (\frac{5}{4}x^{4/5} - 5x^{-1/5})|_{-1}^{0} + (\frac{5}{4}x^{4/5} - 5x^{-1/5}|_0^1)$

$\displaystyle= (0 - \lim_{x\to 0^-} 5x^{-1/5}) - (\frac{5}{4} + 5) + (\frac{5}{4} - 5) - (0 - \lim_{x\to 0^+} 5x^{-1/5})$

$= \infty$

Then $\displaystyle\int_{-1}^{1} \frac{x+1}{\sqrt[5]{x^6}}dx$ diverges

3. (28%)

(a) Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(x+4)^n}{n \cdot 2^n}$

sol.
Ratio Test $\frac{(x+4)^{n+1}}{n+1 \cdot 2^{n+1}} \frac{(x+4)^n}{n \cdot 2^n} = |\frac{x+4}{2}| \cdot |\frac{n}{n+1}| \longrightarrow |\frac{x+4}{2}| < 1$
Then $|x+4| < 2$, that is $-6 < x < -2$
But when $x = -6, \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by Alternating Series Test
when $x = -2, \displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges
So convergent interval is $-6 \leq x < -2$

(b) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin(\frac{1}{\sqrt{n}}) \tan(\frac{1}{\sqrt{n}})$ is convergent or divergent. Give reasons for your answers.
sol.
By Limit Comparison Test
$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}} \sin(\frac{1}{\sqrt{n}}) \tan(\frac{1}{\sqrt{n}})}{n^{-\frac{3}{2}}} = \lim_{n\to\infty} n \sin(\frac{1}{\sqrt{n}}) \tan(\frac{1}{\sqrt{n}}) = \lim_{n\to\infty} \frac{\sin(\frac{1}{\sqrt{n}})}{\frac{1}{\sqrt{n}}} \cdot \frac{\sin(\frac{1}{\sqrt{n}})}{\frac{1}{\sqrt{n}}} \cdot \sec(\frac{1}{\sqrt{n}}) = 1$
Since $\displaystyle\sum_{n=1}^{\infty} n^{-\frac{3}{2}}$ converges, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin(\frac{1}{\sqrt{n}}) \tan(\frac{1}{\sqrt{n}})$ converges

(c) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+1}$ is conditionally convergent, absolutely convergent, or divergent. Give reasons for your answers.
sol.
Since each term $\frac{1}{\sqrt{n}+1}$ is positive and decreasing to zero
By Alternating Series Test we have derive that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+1}$ is convergent
Also, $|\frac{(-1)^n}{\sqrt{n}+1}| = \frac{1}{\sqrt{n}+1} > \frac{1}{\sqrt{n}+\sqrt{n}} = \frac{1}{2\sqrt{n}}$
and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$ is divergent p-series for $p = \frac{1}{2}$
Hence $\displaystyle\sum_{n=1}^{\infty} |\frac{(-1)^n}{\sqrt{n}+1}|$ diverges
Then $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+1}$ is not absolutely convergent
It's conditionally convergent

(d) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos n$ is conditionally convergent, ab-

solutely convergent, or divergent. Give reasons for your answers.
sol.
Since $\lim_{n\ rightarrow\infty} \cos n$ doesn't exist and $\lim_{n\ rightarrow\infty} \cos n \neq 0$
We can not apply by the Alternating Series Test to $\sum_{n=1}^{\infty}(-1)^n \cos n$
Hence $\sum_{n=1}^{\infty}(-1)^n \cos n$ is divergent

4. (10%)

(a) Find $f(x)$ such that $\int (\ln x)^n dx = f(x) - n\int (\ln x)^{n-1} dx$.

(b) Prove that $\int_0^1 (\ln x)^n dx = (-1)^n n!$ for any positive integer n.

sol.

(a) Use integration by parts, $\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx$.
$\therefore f(x) = x(\ln x)^n$.

(b) First we check that $\lim_{b\to 0^+} x(\ln x)^k|_b^1 = 0$ for any finite positive integer k.
$\lim_{b\to 0^+} x(\ln x)^k|_b^1 = 1\cdot(\ln 1)^k - \lim_{b\to 0^+} b(\ln b)^k$. Use L'Hôpital's Rule,

$$\begin{aligned}\lim_{b\to 0^+} b(\ln b)^k &= \lim_{b\to 0^+}\frac{(\ln b)^k}{\frac{1}{b}} = \lim_{b\to 0^+}\frac{k(\ln b)^{k-1}\frac{1}{b}}{-\frac{1}{b^2}} = -k\lim_{b\to 0^+}\frac{(\ln b)^{k-1}}{\frac{1}{b}} = \cdots\\ &= (-1)^{k-1}k!\lim_{b\to 0^+}\frac{(\ln b)}{\frac{1}{b}} = (-1)^{k-1}k!\lim_{b\to 0^+}\frac{\frac{1}{b}}{-\frac{1}{b^2}}\\ &= (-1)^k k!\lim_{b\to 0^+} b = 0.\end{aligned}$$

Next We use mathematical induction to prove the result.
For $n=1$, $\int_0^1 (\ln x)dx = \lim_{b\to 0^+} x(\ln x)|_b^1 - \int_0^1 dx = 0 - 1 = -1$ (o.k.)
Suppose this is true for $n=k$, that is, $\int_0^1 (\ln x)^k dx = (-1)^k k!$.
Then for $n = k+1$,

$$\begin{aligned}\int_0^1 (\ln x)^{k+1} dx &= \lim_{b\to 0^+} x(\ln x)^{k+1}|_b^1 - (k+1)\int_0^1 (\ln x)^k dx\\ &= 0 - (k+1)(-1)^k k! = (-1)^{k+1}(k+1)!\end{aligned}$$

By mathematical induction, this is true for any positive integer n.

5. (12%) Find the area of the surface generated by revolving the curve $y = e^{-2x}$, $x \geq 0$, about the x-axis. sol. 1

$$\text{Area} = \int_0^\infty 2\pi y ds = \int_0^\infty 2\pi y \sqrt{1 + (\frac{dy}{dx})^2} dx$$

$$= \int_0^\infty 2\pi e^{-2x} \sqrt{1 + (-2e^{-2x})^2} dx$$

$$= -\pi \int_1^0 \sqrt{1 + 4u^2} du \qquad (\text{Let } u = e^{-2x})$$

$$= -\frac{\pi}{2} \int_{\tan^{-1}(2)}^0 \sqrt{1 + \tan^2\theta} \sec^2\theta d\theta \qquad (\text{Let } 2u = \tan\theta)$$

$$= \frac{\pi}{2} \int_0^{\tan^{-1}(2)} \sec^3\theta d\theta$$

$$= \frac{\pi}{2} (\frac{1}{2} \sec\theta \tan\theta + \frac{1}{2} \ln|\sec\theta + \tan\theta|) |_0^{\tan^{-1}(2)}$$

$$= \frac{\pi}{4} (2\sqrt{5} + \ln(\sqrt{5} + 2))$$

Here we use

$$\int \sec^3\theta d\theta = \int \sec\theta (\sec^2\theta d\theta)$$

$$= \int \sec\theta d\tan\theta = \sec\theta\tan\theta - \int \tan\theta(\tan\theta\sec\theta d\theta)$$

$$= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta$$

$$\Rightarrow \qquad \int \sec^3\theta d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|)$$

and

$$\sec(\tan^{-1}(2)) = \sqrt{5}$$

sol. 2

$$\text{Area} = \int_0^\infty 2\pi y ds = \int_0^\infty 2\pi e^{-2x} \sqrt{1 + (-2e^{-2x})^2} dx$$

$$= -\pi \int_1^0 \sqrt{1+4u^2}du \qquad (\text{Let } u = e^{-2x})$$

$$= -\pi \int_{\sinh^{-1}(2)}^0 \sqrt{1+\sinh^2\theta}\frac{\cosh\theta}{2}d\theta \qquad (\text{Let } u = \frac{\sinh\theta}{2})$$

$$= \frac{\pi}{2}\int_0^{\sinh^{-1}(2)} \cosh^2\theta d\theta$$

$$= \frac{\pi}{2}\int_0^{\sinh^{-1}(2)} \frac{\cosh 2\theta + 1}{2}d\theta \qquad (\text{"double angle formulas"})$$

$$= \frac{\pi}{2}(\frac{\sinh 2\theta}{4} + \frac{1}{2}\theta)|_0^{\sinh^{-1}(2)}$$

$$= \frac{\pi}{2}(\frac{\sinh\theta\cosh\theta}{2} + \frac{1}{2}\theta)|_0^{\sinh^{-1}(2)} \qquad (\text{"double angle formulas" again})$$

$$= \frac{\pi}{2}(\cosh(\sinh^{-1}(2)) + \frac{1}{2}\sinh^{-1}(2))$$

$$= \frac{\pi}{4}(2\sqrt{5} + \ln(\sqrt{5}+2))$$

$$(\cosh\sinh^{-1}(2) = \sqrt{1+\sinh^2(\sinh^{-1}(2))} = \sqrt{1+4} = \sqrt{5})$$

$$(\text{and } \sinh^{-1}(x) = \ln(x + \sqrt{1+x^2}))$$

6. (10%)

(a) Find the Taylor's series of $\sin(ax) - \tan^{-1} x - x$ at $x = 0$.

(b) Find the value of a for which the limit $\lim\limits_{x\to 0} \dfrac{\sin(ax) - \tan^{-1} x - x}{x^3 + x^4}$ is finite and then evaluate the limit.

sol.

(a) $\sin(ax) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n (ax)^{2n+1}}{(2n+1)!} = ax - \frac{(ax)^3}{3!} + \frac{(ax)^5}{5!} - \dots$

$\tan^{-1} x = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

$\Rightarrow \sin(ax) - \tan^{-1} x - x = (a-2)x + \sum\limits_{n=1}^{\infty} (-1)^n [a^{2n+1} - (2n)!] \frac{x^{2n+1}}{(2n+1)!}$

$= (a-2)x - (a^3 - 2)\frac{x^3}{3!} + (a^5 - 4!)\frac{x^5}{5!} - \dots$

(b) $\lim\limits_{x\to 0} \frac{\sin(ax) - \tan^{-1} x - x}{x^4 + x^3} = \lim\limits_{x\to 0} \frac{a-2}{x^2} - \frac{a^3-2}{3!} + \frac{a^5-4!}{5!} x^2 - \dots \cdot \frac{1}{x+1}$

is finite if $a = 2$

Then $\lim\limits_{x\to 0} \frac{\sin(ax) - \tan^{-1} x - x}{x^4 + x^3} = -\frac{2^3-2}{6} = -1$