1. (18%) Determine whether the given series converges or diverges. Give reasons for your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{\theta^n}{n!}\right)$, where $\theta \neq 0$ is a fixed nonzero constant,

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1+(-1)^n 2n^2}{3n^2}\right)^{\frac{n}{3}}$,

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n+1)(2n+2)\cdots(2n+n)}\left(\frac{27}{4}\right)^n$.

Ans. The series in (a) is ______________________(convergent/divergent) ;

the series in (b) is ______________________(convergent/divergent) ;

the series in (c) is ______________________(convergent/divergent) .

Solution:

(a) Let $a_n = \frac{\theta^n}{n!}$.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{\theta^{n+1}}{(n+1)!}\right|}{\left|\frac{\theta^n}{n!}\right|} = \lim_{n\to\infty} \left|\frac{\theta}{n+1}\right| = 0 < 1,$$

By ratio test, $\sum_{n=1}^{\infty} \left|\frac{\theta^n}{n!}\right|$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{\theta^n}{n!}$ converges.

Let $b_n = \sin\left(\frac{\theta^n}{n!}\right)$

$$\lim_{n\to\infty} \frac{|b_n|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\sin\left(\frac{\theta^n}{n!}\right)\right|}{\left|\frac{\theta^n}{n!}\right|} = 1 \text{ and } \sum |a_n| \text{ converges,}$$

By limit comparison test, $\sum_{n=1}^{\infty} \left|\sin\left(\frac{\theta^n}{n!}\right)\right|$ converges $\Rightarrow \sum_{n=1}^{\infty} \sin\left(\frac{\theta^n}{n!}\right)$ converges.

(b) Let $a_n = (\frac{1+(-1)^n 2n^2}{3n^2})^{\frac{n}{3}}$.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left|\frac{1+(-1)^n 2n^2}{3n^2}\right|^{\frac{1}{3}} = \lim_{n\to\infty} \left|\frac{1}{3n^2} + \frac{2}{3}(-1)^n\right|^{\frac{1}{3}} < 1,$$

By root test, $\sum_{n=1}^{\infty} \left|(\frac{1+(-1)^n 2n^2}{3n^2})^{\frac{n}{3}}\right|$ converges $\Rightarrow \sum_{n=1}^{\infty} (\frac{1+(-1)^n 2n^2}{3n^2})^{\frac{n}{3}}$ converges

(c) Let $a_n = \frac{n!}{(2n+1)(2n+2)\cdots(2n+n)}\left(\frac{27}{4}\right)^n = \frac{n!(2n)!}{(3n)!}\left(\frac{27}{4}\right)^n$, $a_n > 0$,$\forall n$.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!(2n+2)!}{(3n+3)!}\left(\frac{27}{4}\right)^{n+1}}{\frac{n!(2n)!}{(3n)!}\left(\frac{27}{4}\right)^n} = \lim_{n\to\infty} \frac{(n+1)(2n+1)(2n+2)}{(3n+1)(3n+2)(3n+3)} \cdot \frac{27}{4} = 1$$

We cannot decide from the Ratio Test whether the series converges.

Notice that $\frac{a_{n+1}}{a_n} = \frac{(n+1)(2n+1)(2n+2)}{(3n+1)(3n+2)(3n+3)} \cdot \frac{27}{4} = \frac{2(n+1)(2n+1)}{3(3n+1)(3n+2)} \cdot \frac{27}{4} = \frac{n^2+\frac{3}{2}n+\frac{1}{2}}{n^2+n+\frac{2}{9}} > 1,$

then a_{n+1} is always greater than a_n,

Therefore all terms are greater than $a_1 = \frac{9}{4}$, and $a_n \nrightarrow 0$. The series diverges.

2. (10%) Find the radius and interval of convergence of the power series
$p(x) = \sum_{k=2}^{\infty} \frac{3^k}{k \ln k}(2x-1)^k$. For what values of x does the series converge absolutely/conditionally?

Ans. (a) The radius of convergence is ________________________________ ;

(b) the interval of convergence is ________________________________ ;

(c) $\{x \,|\, p(x) \text{ converges absolutely}\} =$ ________________________________ ;

(d) $\{x \,|\, p(x) \text{ converges conditionally}\} =$ ________________________________ .

Solution:

Let $a_k = \frac{3^k}{k \ln k}(2x-1)^k$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\frac{3^{k+1}}{(k+1)\ln(k+1)}(2x-1)^{k+1}}{\frac{3^k}{k\ln k}(2x-1)^k}\right| = |3(2x-1)|$$

If $|3(2x-1)| < 1$,$p(x)$ is convergence absolutely $\Rightarrow |x - \frac{1}{2}| < \frac{1}{6} \Rightarrow R = \frac{1}{6}$, $\frac{1}{3} < x < \frac{2}{3}$

$*$ If $x = \frac{1}{3}$, then $p(\frac{1}{3}) = \sum_{k=2}^{\infty} \frac{3^k}{k\ln k}(-\frac{1}{3})^k = \sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}$ is an alternating series.

$\sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln k}$ converges, since $\frac{1}{k\ln k} > \frac{1}{(k+1)\ln(k+1)} > 0 \ \forall k \geq 2$, and $\lim_{k\to\infty} \frac{1}{k\ln k} = 0$.

But $\sum_{k=2}^{\infty} \left|\frac{(-1)^k}{k\ln k}\right| = \sum_{k=2}^{\infty} \frac{1}{k\ln k}$ diverges by integral test ($\because \int_2^{\infty} \frac{dx}{x\ln x} = \infty$).

Therefore $p(\frac{1}{3})$ converges conditionally.

$*$ If $x = \frac{2}{3}$, then $p(\frac{2}{3}) = \sum_{k=2}^{\infty} \frac{3^k}{k\ln k}(\frac{1}{3})^k = \sum_{k=2}^{\infty} \frac{k}{k\ln k}$ diverges by integral test.

So

(a) The radius of convergence is $\frac{1}{6}$.
(b) the interval of convergence is $[\frac{1}{3}, \frac{2}{3})$
(c) $\{x \,|\, p(x) \text{ converges absolutely}\} = (\frac{1}{3}, \frac{2}{3})$
(d) $\{x \,|\, p(x) \text{ converges conditionally}\} = \{\frac{1}{3}\}$

3. (12%) Represent $\sin^2 x$ as the sum of its Taylor series centered at $\frac{\pi}{3}$.

Ans. $\sin^2 x =$ ______________________________ .

<u>Solution:</u>

Method1:

$f(x) = \sin^2 x, f'(x) = 2\sin x\cos x = \sin 2x, f''(x) = 2\cos 2x, \ldots,$

$f^{(2n)}(x) = (-1)^{n+1}2^{2n-1}\cos 2x, f^{(2n+1)}(x) = (-1)^n 2^{2n}\sin 2x \ldots$

$f(\frac{\pi}{3}) = \frac{3}{4}, f'(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}, f''(\frac{\pi}{3}) = -1, \ldots, f^{(2n)}(\frac{\pi}{3}) = (-1)^n \cdot 2^{2n-2}, f^{(2n+1)}(\frac{\pi}{3}) = (-1)^n 2^{2n-1}\sqrt{3}, \ldots$

$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{3})}{n!}(x-\frac{\pi}{3})^n = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{2^{k-1}\cos(\frac{2}{3}\pi + \frac{k}{2}\pi)}{n!}(x-\frac{\pi}{3})^n$

Method2:

$$\begin{aligned}
\sin^2 x &= \frac{1-\cos 2x}{2} \\
&= \frac{1}{2} - \frac{1}{2}\cos[2(x-\frac{\pi}{3}) + \frac{2}{3}\pi] \\
&= \frac{1}{2} - \frac{1}{2}[\cos\frac{2}{3}\pi\cos 2(x-\frac{\pi}{3}) - \sin\frac{2}{3}\pi\sin 2(x-\frac{\pi}{3})] \\
&= \frac{1}{2} + \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}(x-\frac{\pi}{3})^{2n} + \frac{\sqrt{3}}{4}\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}}{(2n+1)!}(x-\frac{\pi}{3})^{2n+1} \\
&= \frac{3}{4} + \sum_{n=1}^{\infty}\frac{2^{k-1}\cos(\frac{2}{3}\pi + \frac{k}{2}\pi)}{n!}(x-\frac{\pi}{3})^n
\end{aligned}$$

4. (10%) Evaluate the sum of the infinite series $\frac{1^2}{0!}+\frac{2^2}{1!}+\frac{3^2}{2!}+\frac{4^2}{3!}+\cdots+\frac{(n+1)^2}{n!}+\cdots=S$.

Ans. $S =$ ____ .

Solution:

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$x \cdot e^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$

$\frac{d}{dx}(x \cdot e^x) = (1+x)e^x = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}$

$(x+x^2)e^x = \sum_{n=0}^{\infty} \frac{(n+1)x^{n+1}}{n!}$

$\frac{d}{dx}[(x+x^2)e^x] = (1+3x+x^2)e^x = \sum_{n=0}^{\infty} \frac{(n+1)^2x^n}{n!}$

Let $x = 1 \Rightarrow S = \sum_{n=0}^{\infty} \frac{(n+1)^2}{n!} = 5e.$

5. (10%) Assume that all the first and the second partial derivatives of $u(x,y)$ are continuous and that $u(x,y)$ satisfies $\frac{\partial^2 u}{\partial x \partial y} \equiv 0$. Let a, b be two constants and $f(x,y) = u(x,y)e^{ax+by}$. Find the values of a, b such that $\frac{\partial^2 f}{\partial x \partial y} - 2\frac{\partial f}{\partial x} - 3\frac{\partial f}{\partial y} + abf \equiv 0$.

Ans. $a =$ ____________________ , $b =$ ____________________ .

Solution:

$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}e^{ax+by} + u(x,y) \cdot ae^{ax+by}$

$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}e^{ax+by} + u(x,y) \cdot be^{ax+by}$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}e^{ax+by} + \frac{\partial u}{\partial x} \cdot be^{ax+by} + \frac{\partial u}{\partial y} \cdot ae^{ax+by} + u(x,y) \cdot abe^{ax+by}$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}e^{ax+by} + \frac{\partial u}{\partial y} \cdot ae^{ax+by} + \frac{\partial u}{\partial x} \cdot be^{ax+by} + u(x,y) \cdot abe^{ax+by}$

$\because \frac{\partial^2 f}{\partial x \partial y} - 2\frac{\partial f}{\partial x} - 3\frac{\partial f}{\partial y} + abf \equiv 0$

$\therefore \frac{\partial^2 u}{\partial y \partial x}e^{ax+by} + \frac{\partial u}{\partial x} \cdot be^{ax+by} + \frac{\partial u}{\partial y} \cdot ae^{ax+by} + u(x,y) \cdot abe^{ax+by} - 2 \cdot [\frac{\partial u}{\partial x}e^{ax+by} + u(x,y) \cdot ae^{ax+by}]$

$-3 \cdot [\frac{\partial u}{\partial y}e^{ax+by} + u(x,y) \cdot be^{ax+by}] + ab \cdot u(x,y)e^{ax+by}$

$= \frac{\partial u}{\partial x}e^{ax+by}(b-2) + \frac{\partial u}{\partial y}e^{ax+by}(a-3) + u(x,y)e^{ax+by}(2ab - 2a - 3b) \equiv 0.$

$\Rightarrow b = 2, a = 3.$

6. (15%) Let $F(x,y,z) = x^2 + 2z + \int_y^z \sqrt[3]{t^2+1-y^2}\,dt$.

(a) Find the plane tangent to the surface $\{(x,y,z) \in \mathbb{R}^3 \mid F(x,y,z) = 2\}$ at $(2,-1,-1)$.

(b) Let $z = z(x,y)$ be the function implicitly defined by $F(x,y,z) = 2$ around $(2,-1,-1)$. Find the direction(s) at the point $(x,y) = (2,-1)$ along which $z(x,y)$ increases most rapidly. That is, find unit vector $\mathbf{u} \in \mathbb{R}^2$ such that $(D_{\mathbf{u}}z)(2,-1)$ attains its maximum.

(c) Evaluate $\frac{\partial^2 z}{\partial x \partial y}$ at $(x,y,z) = (2,-1,-1)$ for the function $z(x,y)$ given in (b).

Ans. (a) The equation of the tangent plane is ____________________________ ;

(b) the direction(s) is(are) ____________________________ ;

(c) At $(x,y,z) = (2,-1,-1)$, $\frac{\partial^2 z}{\partial x \partial y} =$ ____________________________ .

Solution:

(a) $\nabla F(x,y,z) = (2x, \int_y^z (\frac{1}{3})(t^2+1-y^2)^{-\frac{2}{3}})(-2y)dt - \sqrt[3]{y^2+1-y^2}, 2 + \sqrt[3]{z^2+1-y^2})$

$\nabla F(2,-1,-1) = (4,-1,2+1) = (4,-1,3)$

tangent plane: $(4,-1,3) \cdot (x-2, y+1, z+1) = 0 \Rightarrow 4x - y + 3z = 6$.

(b) $z(x,y)$ is differential at $(2,-1)$ and $D_{\mathbf{u}}z = \nabla z \cdot \mathbf{u} \le |\nabla z|$

when $\mathbf{u} = \frac{\nabla z}{|\nabla z|}$, $D_{\mathbf{u}}z$ '=' $|\nabla z|$

$F(x,y,z(x,y)) = 2$

$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$, $\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$

At $(2,-1)$, $\nabla z = (-\frac{4}{3}, -\frac{-1}{3}) = (-\frac{4}{3}, \frac{1}{3})$, $|\nabla z| = \sqrt{\frac{16+1}{9}} = \frac{\sqrt{17}}{3}$, $\mathbf{u} = (-\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}})$.

(c) $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(-\frac{2x}{2+\sqrt[3]{z^2+1-y^2}}\right) = 2x \cdot \frac{\frac{1}{3}(x^2+1-y^2)^{-\frac{2}{3}} \cdot \{2z \cdot \frac{\partial z}{\partial y} - 2y\}}{(2+\sqrt[3]{z^2+1-y^2})^2}$

At $(2,-1,-1)$, $\frac{\partial^2 z}{\partial x \partial y} = 2 \cdot 2 \cdot \frac{\frac{1}{3} \cdot 1 \cdot \{2(-1) \cdot \frac{1}{3} - 2(-1)\}}{3^2} = \frac{4}{27}\{2 - \frac{2}{3}\} = \frac{16}{81}$.

7. (12%) Find all the local maxima, local minima, and saddle points of $G(x,y) = x^3 - x^2 + 3x^2y - 4xy - 3y^3 + \frac{13}{2}y^2$.

Ans. G has a local maximum = ____________ at $(x,y) =$ ____________ ;

G has a local minimum = ____________ at $(x,y) =$ ____________ ;

G has saddle point(s) at $(x,y) =$ ________________________ .

Solution:

$$G_x = 3x^2 - 2x + 6xy - 4y$$

$$G_y = 3x^2 - 4x - 9y^2 + 13y$$

$$\begin{cases} G_{xx} = 6x - 2 + 6y \\ G_{xy} = G_{yx} = 6x - 4 \\ G_{yy} = -18y + 13 \end{cases}$$

$$\Rightarrow \text{Hession of } G = \begin{vmatrix} 6x - 2 + 6y & 6x - 4 \\ 6x - 4 & -18y + 13 \end{vmatrix}$$

解 $\begin{cases} G_x = 0 \\ G_y = 0 \end{cases} \Rightarrow (x,y) = (0,0), (14,-7), (\frac{2}{3}, \frac{1}{9}), (\frac{2}{3}, \frac{4}{3})$

代入 Hession 判別: $\Rightarrow (0,0)$, $(14,-7)$, $(\frac{2}{3}, \frac{4}{3})$ 都是 saddle point

$\Rightarrow (\frac{2}{3}, \frac{1}{9})$ 是 minimum, $G(\frac{2}{3}, \frac{1}{9}) = -\frac{109}{486}$.

8. (13%) Find the extreme values of $H(x,y,z) = xy + z^3$ on the intersection of the plane $x - y = 0$ and the ellipsoid $x^2 + 2y^2 + 3z^2 = 9$.

Ans. The maximum of H is ______________ at $(x,y,z) =$ ______________ ;

the minimum of H is ______________ at $(x,y,z) =$ ______________ .

Solution:

$\nabla H = (y, x, 3z^2)$

$\nabla(x - y) = (1, -1, 0)$

$\nabla(x^2 + 2y^2 + 3z^2) = (2x, 4y, 6z)$

$\exists\ \lambda,\ \mu$, s.t.

$$\begin{cases} y = \lambda + \mu \cdot 2x & \cdots (1) \\ x = -\lambda + \mu \cdot 4y & \cdots (2) \\ 3z^2 = 0 + \mu \cdot 6z & \cdots (3) \\ x - y = 0 & \cdots (4) \\ x^2 + 2y^2 + 3z^2 = 9 & \cdots (5) \end{cases}$$

By (4): $x = y$, (1)+(2): $2x = 6x\mu$, $x = y = 0$ or $\mu = \frac{1}{3}$

Case1:

If $x = 0$, $y = 0$, $z = \pm\sqrt{3}$, $H = \pm 3\sqrt{3}$

Case2:

If $\mu = \frac{1}{3}$ plug in (3): $3z^2 = 2z \Rightarrow z = 0$, $z = \frac{2}{3}$

If $z = 0$, $x = y$ plug in (5): $3x^2 = 9 \Rightarrow x = y = \pm\sqrt{3}$,$H = 3$.

If $z = \frac{2}{3}$, $x = y$ plug in (5): $3x^2 = \frac{23}{3} \Rightarrow x = y = \pm\frac{\sqrt{23}}{3}$, $H = \frac{77}{27} < 3$.

$\therefore$ The maximum of H is $3\sqrt{3}$ at $(0, 0, \sqrt{3})$, the minimum of H is $-3\sqrt{3}$ at $(0, 0, -\sqrt{3})$