95下 微積分甲統一教學一組期末考題暨參考答案

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1. (20%) Consider the transformation $u(x,y) = x+y, v(x,y) = \dfrac{x}{y}$, which maps $\mathbb{R}^2_+ = \{(x,y), x>0, y>0\}$ onto $\mathbb{R}^2_+ = \{(u,v), u>0, v>0\}$.

   (a) Find the inverse of the above transformation. That is, solve (x,y) in terms of (u,v).

   (b) Evaluate the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

   (c) Evaluate the double integral $I_2 = \displaystyle\iint_{\mathbb{R}^2_+} \left(\frac{x+y}{y}\right)^{\frac{3}{5}} e^{-2x-2y}\, dA$.

Ans. $x =$ ________________ , $y =$ ________________ ,

$\dfrac{\partial(x,y)}{\partial(u,v)} =$ ________________ , $I_2 =$ ________________ .

Solution:

(a) $\begin{cases} u = x+y \\ v = \frac{x}{y} \end{cases} \Rightarrow \begin{cases} x = \frac{uv}{1+v} \\ y = \frac{u}{1+v} \end{cases}$

(b) $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{v}{1+v} & \frac{1}{1+v} \\ \frac{u}{(1+v)^2} & -\frac{u}{(1+v)^2} \end{vmatrix} = -\frac{uv}{(1+v)^3} - \frac{u}{(1+v)^3} = -\frac{u}{(1+v)^2}$

(c)

$$\begin{aligned}
I_2 &= \int_0^\infty \int_0^\infty (v+1)^{\frac{3}{5}} e^{-2u} \cdot \frac{u}{(1+v)^2}\, dudv \\
&= \int_0^\infty (v+1)^{-\frac{7}{5}}\, dv \cdot \int_0^\infty u \cdot e^{-2u}\, du \\
&= (-\frac{5}{2}(v+1)^{-\frac{2}{5}} \mid_0^\infty)(-\frac{1}{2}\int_0^\infty u\, de^{-2u}) \\
&= -\frac{5}{2} \cdot (-\frac{1}{2})(u \cdot e^{-2u} - (-\frac{1}{2} e^{-2u}) \mid_0^\infty) = \frac{5}{4} \cdot \frac{1}{2} = \frac{5}{8}
\end{aligned}$$

2. (20%) Consider the vector field

$$\mathbf{F} = \left(\frac{2xyz}{x^2y+z} + y(\cos x)e^{y\sin x} \,,\, \frac{x^2z}{x^2y+z} + (\sin x)e^{y\sin x} \,,\, \frac{z}{x^2y+z} + \ln(x^2y+z) \right),$$

which is defined on $D_+ = \{(x,y,z), x>0, y>0, z>0\}$.

(a) Find a potential function $f(x,y,z)$ of $\mathbf{F}$ if $\mathbf{F}$ is conservative on D_+, or show that $\mathbf{F}$ is not conservative on D_+.

(b) Evaluate the line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the line segment from $(\pi,1,1)$ to $\left(\frac{\pi}{2},8,2\right)$.

Ans. (a) $\mathbf{F}$ is conservative and $f(x,y,z) =$ ____________________________ ,

or $\mathbf{F}$ is not conservative because ______________________________________ ,

(b) $\int_C \mathbf{F}\cdot\mathbf{T}\,ds =$ ____ .

Solution:

(a) Let $\mathbf{F} = (M,N,P)$ $\mathbf{F}$ is conservative $\Leftrightarrow \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}, \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}$

$$\frac{\partial N}{\partial x} = \frac{2xz(x^2y+z) - x^2z(2xy)}{(x^2y+z)^2} + cosx \cdot e^{y\sin x} + \sin x\cdot y\cos x\cdot e^{y\sin x} = \frac{\partial M}{\partial y}$$

$$\frac{\partial P}{\partial y} = \frac{x^4y}{(x^2y+z)^2} = \frac{\partial N}{\partial z}$$

$$\frac{\partial M}{\partial z} = \frac{2x^3y^2}{(x^2y+z)^2} = \frac{\partial P}{\partial x}$$

$\therefore$ $\mathbf{F}$ is conservative

$$f = \int M\,dx = \int \frac{2xyz}{x^2y+z} + y(\cos x e^{y\sin x})\,dx$$

$$= z\int \frac{2xy}{x^2y+z}\,dx + \int e^{y\sin x}\,d(y\sin x) = z\cdot\ln(x^2y+z) + e^{y\sin x} + h(y,z)$$

$$\frac{\partial f}{\partial y} = \frac{x^2z}{x^2y+z} + \sin x\cdot e^{y\sin x} + \frac{\partial h}{\partial y} \Rightarrow \frac{\partial h}{\partial y} = 0, \text{ so we let } h(y,z) = g(z)$$

$$\frac{\partial f}{\partial z} = \ln(x^2y+z) + z\cdot\frac{1}{x^2y+z} \Rightarrow g'(z) = 0$$

$$\Rightarrow f(x,y,z) = z\ln(x^2y+z) + e^{y\sin x} + C.$$

(b) $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = F(\frac{\pi}{2},8,2) - F(\pi,1,1) = 2\ln 2 + \ln(\pi^2+1) + e^8 - 1.$

3. (12%) Given $a > 0$, consider the iterated integral

$$I(a) = \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} ze^{-(x^2+y^2+z^2)}\, dz\, dy\, dx.$$

By Fubini's theorem $I(a) = \iiint_{D(a)} ze^{-(x^2+y^2+z^2)}\, dV$. Specify the region of integration $D(a)$, and evaluate $I(a)$.

Ans. $D(a) =$ ______________________________ ,

$I(a) =$ ______________________________ .

Solution:

(a) $D(a) = \{(x, y, z) \mid x^2 + y^2 + z^2 \le a^2, x > 0, z > 0\}$.

(b)

$$\begin{aligned}
I(a) &= \int_0^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^a \rho\cos\phi \cdot e^{-\rho^2} \cdot \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \\
&= \int_0^a \rho^3 e^{-\rho^2}\, d\rho \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \cdot \int_0^{\frac{\pi}{2}} \cos\phi \sin\phi\, d\phi \\
&= \frac{1}{2}(-\rho^2 e^{-\rho^2} - e^{-\rho^2}) \,\Big|_0^a \cdot \pi \cdot (\frac{\sin^2\phi}{2}) \,\Big|_0^{\frac{\pi}{2}} \\
&= \frac{1}{2}(-a^2 e^{-a^2} - e^{-a^2} + 1) \cdot \pi \cdot \frac{1}{2} \\
&= \frac{\pi}{4}(1 - a^2 e^{-a^2} - e^{-a^2})
\end{aligned}$$

4. (15%) Apply Green's theorem to find the area of the region bounded by the curve parametrized by $\mathbf{r}(t) = (\cos t, \sin^3 t)$, $0 \le t \le 2\pi$.

Ans. The area is ____________________________________ .

Solution:

By Green's theorem, Let $M = -y$, $N = x$.

$$
\begin{aligned}
\iint_D dA &= \frac{1}{2}\int_C -y\,dx + x\,dy \\
&= \frac{1}{2}\int_0^{2\pi} -\sin^3 t\, d\cos t + \cos t\, d\sin^3 t \\
&= \frac{1}{2}\int_0^{2\pi} \sin^4 t + 3\sin^2 t\cos^2 t\, dt \\
&= \frac{1}{2}\int_0^{2\pi} 3\sin^2 t - 2\sin^4 t\, dt \\
&= \frac{1}{2}\int_0^{2\pi} (3\cdot\frac{1-\cos 2t}{2} - \frac{1}{2} + \cos 2t - \frac{1+\cos 4t}{4})\, dt \\
&= \frac{1}{8}\int_0^{2\pi} 3 - 2\cos 2t - \cos 4t\, dt \\
&= \frac{1}{8}\cdot 6\pi = \frac{3\pi}{4}
\end{aligned}
$$

5. (15%) Evaluate $I_5 = \oint_\Gamma (y + \sin x)\, dx + (z^2 + \cos y)\, dy + x^3\, dz$, where Γ is the closed curve $\mathbf{r}(t) = (\sin t, \cos t, \sin 2t)$, $0 \le t \le 2\pi$, oriented by increasing t.

Hint. You may either evaluate the line integral directly, or apply Stokes' theorem. For the latter approach, use the relation $\sin 2t = 2 \sin t \cos t$ to find a surface on which the curve Γ lies.

Ans. $I_5 =$ ____ .

Solution:

Let $\mathbf{F} = (y + \sin x, x^2 + \cos y, x^3)$ $\nabla \times \mathbf{F} = -2z\mathbf{i} - 3x^2\mathbf{j} - \mathbf{k}$

$$\begin{aligned}
I_5 &= \int \mathbf{F}\, d\mathbf{r} \\
&= \int_0^{2\pi} (\cos t + \sin(\sin t), \sin^2(2t) + \cos(\cos t), \sin^3 t) \cdot (\cos t, -\sin t, 2\cos 2t)\, dt \\
&= \int_0^{2\pi} \cos^2 + \cos t \cdot \sin(\sin t) - \sin t \sin^2 2t - \sin t \cdot \cos(\cos t) + 2\cos 2t \cdot \sin^3 t\, dt \\
&= \int_0^{2\pi} \cos^2 + \cos t \cdot \sin(\sin t) - \sin t \cdot \cos(\cos t) \\
&\qquad - \sin t \cdot (4\sin^2 t \cos^2 t) + 2(2\cos^2 t - 1) \cdot \sin^3 t\, dt \\
&= \int_0^{2\pi} \cos^2 + \cos t \cdot \sin(\sin t) - \sin t \cdot \cos(\cos t) - 2\sin^3 t\, dt \\
&= \frac{1 + \sin 2t}{2} - \cos(\sin t) + \sin(\cos t) + 2\cos t - \frac{2}{3}\cos^3 t \,\Big|_0^{2\pi} \\
&= \pi
\end{aligned}$$

6. (18%) Let $a > 0, b, c \in \mathbb{R}, b < c$, and S be the part of the cylinder $S = \{(x, y, z), x^2 + y^2 = a^2, b \leq z \leq c\}$, which is oriented with the unit normal pointing **toward** the z-axis. Evaluate the surface integral $I_6 = \iint\limits_S x^3\, dydz + x^2y\, dzdx + x^2z\, dxdy$.

Ans. $I_6 =$ ____ .

Solution:

Let $\mathbf{F} = x^3\mathbf{i} + x^2y\mathbf{j} + x^2z\mathbf{k}$, $\text{div}\mathbf{F} = 3x^2 + x^2 + x^2 = 5x^2$.
By divergence theorem

$$\iiint\limits_I 5x^2\, dxdydz = -I_6 + \iint\limits_{x^2+y^2\leq a, z=c} x^2z\, dxdy - \iint\limits_{x^2+y^2\leq a^2, z=b} x^2z\, dxdy.$$

$$\begin{aligned}
I_6 &= -\int_b^c \int_0^{2\pi} \int_0^a 5r^2\cos^2\theta \cdot r\, drd\theta dz + \int_0^{2\pi} \int_0^a cr^2\cos^2\theta \cdot r\, drd\theta \\
&\quad - \int_0^{2\pi} \int_0^a br^2\cos^2\theta \cdot r\, drd\theta \\
&= \frac{5a^4\pi}{4}(b - c) + \frac{a^4\pi}{4}c - \frac{a^4\pi}{4}b \\
&= \pi a^4(b - c)
\end{aligned}$$