95 上微積分甲統一教學一組

期中考參考答案

1. (20%)

（a）Given $\lim_{x\to 2}\frac{x-2}{\sqrt{px+q}-3}=9$, find p and q ;

（b）Evaluate $\lim_{x\to 0}\frac{\int_0^{2x}\tan t\,dt-2x^2}{x^4}=L_2$;

（c）Find $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\sin^2\frac{i\pi}{2n}=L_3$.

Ans （a）p＝____________, q＝______________ 。

（b）L_2 ＝____________ 。

（c）L_3 ＝____________ 。

....

Sol：

（a）$\lim_{x\to 2}\sqrt{px+q}=3\ \Rightarrow\ \sqrt{2p+q}=3\ \Rightarrow\ 2p+q=9$

$$\lim_{x\to 2}\frac{x-2}{\sqrt{px+q}-3}=\lim_{x\to 2}\frac{(x-2)(\sqrt{px+q}+3)}{px+q-9}=\lim_{x\to 2}\frac{(x-2)(\sqrt{px+q}+3)}{p(x-2)}=\lim_{x\to 2}\frac{\sqrt{px+q}+3}{p}$$

$$=\frac{\sqrt{2p+q}+3}{p}=\frac{\sqrt{9}+3}{p}=9\quad\Rightarrow\begin{cases}\text{p}=\dfrac{2}{3}\\ \text{q}=\dfrac{23}{3}\end{cases}$$

（b）$$L_2=\lim_{x\to 0}\frac{\int_0^{2x}\tan t\,dt-2x^2}{x^4}=\lim_{x\to 0}\frac{2\tan 2\text{x}-4x}{4x^3}=\lim_{x\to 0}\frac{4\sec^2 2\text{x}-4}{12x^2}$$

$$=\lim_{x\to 0}\frac{1-\cos^2 2\text{x}}{3x^2\cos^2 2\text{x}}=\lim_{x\to 0}\frac{\sin^2 2\text{x}}{3x^2\cos^2 2\text{x}}=\lim_{x\to 0}\frac{4}{3}\frac{\sin^2 2\text{x}}{(2x)^2}\frac{1}{\cos^2 2\text{x}}=\frac{4}{3}$$

（c）$$L_3=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\sin^2\frac{i\pi}{2n}=\pi\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sin^2(\frac{\pi}{2}\cdot\frac{i}{n})=\pi\int_0^1\sin^2\frac{\pi x}{2}dx=\pi\int_0^1\frac{1-\cos\pi x}{2}dx=\frac{\pi}{2}$$

2. (10%)

Find the equation of the line normal to the graph of $\sin(xy) = x^2 \cos y$ at the point $(2, \frac{\pi}{2})$.

Ans The equation of the line：______________________________。

....

Sol：

$$\sin(xy) = x^2 \cos y$$

$$\frac{d}{dx}\sin(xy) = \frac{d}{dx}(x^2 \cos y)$$

$$\cos(xy)(y + x\frac{dy}{dx}) = 2x\cos y - x^2 \frac{dy}{dx}\sin y$$

$$\frac{dy}{dx} = \frac{2x\cos y - y\cos(xy)}{x\cos(xy) + x^2 \sin y} \Rightarrow \left.\frac{dy}{dx}\right|_{(2,\frac{\pi}{2})} = \frac{\pi}{4}$$

Normal Line： $y - \frac{\pi}{2} = -\frac{4}{\pi}(x-2)$

3. (15%)

A square paper measures $5\sqrt{2}$ cm by $5\sqrt{2}$ cm. A pyramid is created by removing the four congruent shaded triangles shown below, and then folding along the dotted lines. The base of the pyramid is a square measuring x cm by x cm.

(a) Let V(x) be the volume of this pyramid. Find V(x).

(b) Find the maximum possible volume of the pyramid and the value of x for which it occurs.

Ans (a) V(x)=____________________,

(b) Maximum volume=______________ at x=_____________.

....

Sol：

$$h = \sqrt{(\frac{10-x}{2})^2 - (\frac{x}{2})^2} = \sqrt{25-5x} \Rightarrow V(x) = \frac{1}{3}x^2\sqrt{25-5x}$$

$$\frac{dV(x)}{dx} = \frac{2}{3}x\sqrt{25-5x} - \frac{5x^2}{6\sqrt{25-5x}} = 0 \Rightarrow x = 4 \text{ or } 0$$

$V(4) = \frac{16}{3}\sqrt{5}$, $V(0) = 0.$ $\Rightarrow$ x=4 has the maximum volume $\frac{16}{3}\sqrt{5}\ cm^2$

4. (15%)

(a) State the mean value theorem (differential form.) No proof is needed.

Apply mean value theorem to solve (b) and (c).

(b) Evaluate $\lim_{n\to\infty} \sin((x+2)^{\frac{3}{4}}) - \sin(x^{\frac{3}{4}}) = L_4$

Ans $L_4 =$ ________________.

(c) Prove the inequalities $\frac{1}{25} < \sqrt[3]{66} - 4 < \frac{1}{24}$.

....

Sol:

(a) Suppose f is continuous on [a, b] and differentiable on (a, b).

Then there is at least one point C in (a, b) at which $f(b) - f(a) = f'(c)(b-a)$.

(b) Let $f(z) = \sin(z^{3/4})$, then $f'(z) = \cos(z^{3/4}) \cdot \frac{3}{4} \cdot z^{-1/4}$

Let a=x, b=x+2, the Mean Value Theorem shows that:

$\exists\, c \in (a,\ b)\ s.t.\ f(b) - f(a) = f'(c)(b-a)$

$\sin(x+2)^{3/4} - \sin(x^{3/4}) = \cos(c^{3/4}) \cdot \frac{3}{4} \cdot c^{-1/4} \cdot 2$

As $x \to \infty$ then $c \to \infty$, $\lim_{x\to\infty} \sin(x+2)^{3/4} - \sin(x^{3/4}) = \lim_{c\to\infty} \cos(c^{3/4}) \cdot \frac{3}{4} \cdot c^{-1/4} \cdot 2$

$\because |\cos(c^{3/4})| \le 1$ and $\lim_{c\to\infty} c^{-1/4} = 0$

By the Sandwich theorem, we have $\lim_{x\to\infty} \sin(x+2)^{3/4} - \sin(x^{3/4}) = 0$

(c) $Let\ g(x) = \sqrt[3]{x},\ g'(x) = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{x^2}}$

By Mean Value Theorem, $\exists\, c \in (64,\ 66)\ s.t.\ g'(c) = \frac{\sqrt[3]{66} - 4}{2}$

and g'(x) is decreasing on (0, ∞)

$1^\circ\ c > 64 \Rightarrow g'(c) < g'(64) = \frac{1}{48} \Rightarrow \sqrt[3]{66} - 4 < \frac{1}{24}$.

$2^\circ\ c < 66 < (\sqrt{50/3})^3,\ \frac{\sqrt[3]{66} - 4}{2} > g'((\sqrt{50/3})^3) = \frac{1}{50} \Rightarrow \sqrt[3]{66} - 4 > \frac{1}{25}$.

5. (20%) Study the function $y = f(x) = \dfrac{(x-2)^2}{x+1}$ and answer the following questions.

（1）The domain of $y = f(x)$ is ______________________________.

（2）$f'(x) =$ ______________________________.

（3）$y = f(x)$ has critical point(s) at x＝______________________________.

（4）$y = f(x)$ is increasing on intervals ______________________________.

$y = f(x)$ is decreasing on intervals ______________________________.

（5）$f''(x) =$ ______________________________.

（6）$y = f(x)$ is concave upward on intervals ______________________________.

$y = f(x)$ is concave down on intervals ______________________________.

（7）Find the (x, y)-coordinates of the following points if exist.

local maximum point(s)：______________________________.

local minimum point(s)：______________________________.

inflection point(s)：______________________________.

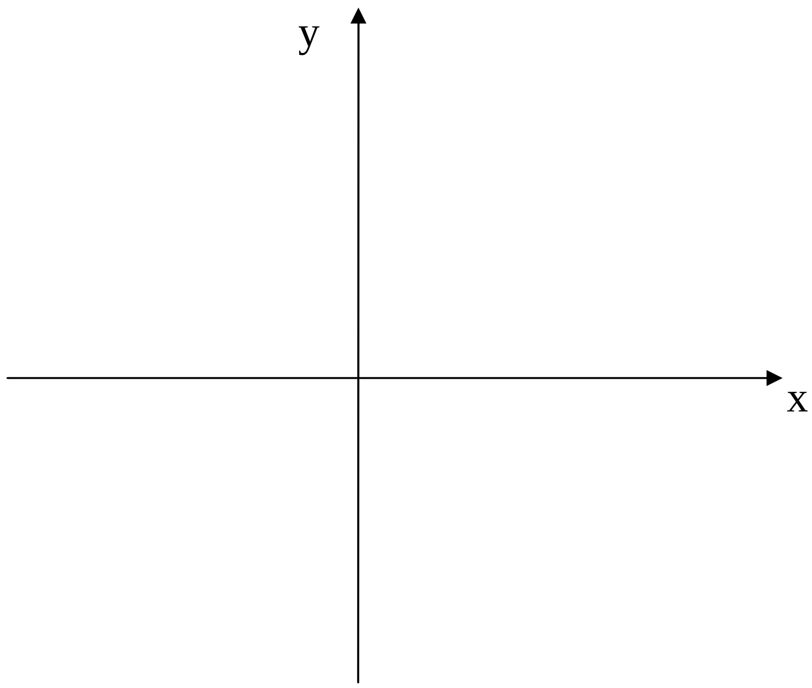
（8）Find the asymptotes of the graph $y = f(x)$ if exist.

Vertical asymptotes(s)：______________________________.

Horizontal asymptotes(s)：____________________ as x→__________.

Slanted asymptotes(s)：____________________ as x→__________.

（9）Sketch the graph of $y = f(x)$ below.



（1） R\{－1} or (－∞,－1)∪(－1,∞).

（2） $f\,'(x)=\dfrac{(x-2)(x+4)}{(x+1)^2}$

（3） $x=-4,\ 2$

（4） $y=f(x)$ is increasing on intervals (－∞,－4), (2,∞)

$y=f(x)$ is decreasing on intervals (－4,－1), (－1,2)

（5） $f\,''(x)=\dfrac{18}{(x+1)^3}$

（6） $y=f(x)$ is concave upward on intervals (－1,∞)

$y=f(x)$ is concave downward on intervals (－∞,－1)

（7） local maximum point(s)：(－4, －12).

local minimum point(s)：(2, 0).

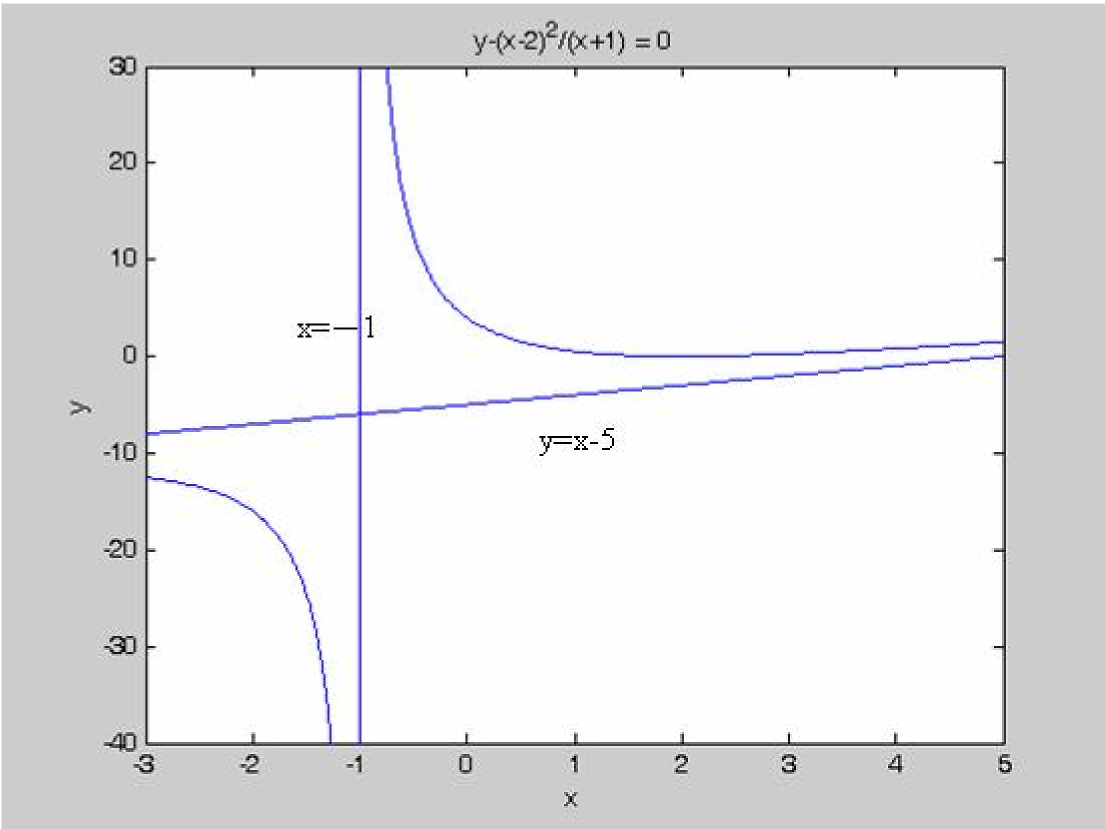
No inflection points!

（8） Vertical asymptotes(s)： x＝－1 is a vertical asymptote.

No Horizontal asymptotes.

Slanted asymptotes(s)：y＝x+5 as x→ ±∞ .

（9）



6. (10%) Given two positive constants a, b, $b > a > 0$, the region bounded by the curve $x^2 + (y-b)^2 = a^2$ is revolved about the x-axis to generate a solid. Find the volume of this solid of revolution. (Certainly, you can use without proof the formula of the area of circles.)

Ans. The volume is ____.

....

Sol：

$$\left.\begin{aligned} y &= b + \sqrt{a^2 - x^2} \\ y &= b + \sqrt{a^2 - x^2} \end{aligned}\right\} \; -a \le x \le a .$$

Disc Method：

$$V = \int_{-a}^{a} \pi(b + \sqrt{a^2 - x^2})^2 - \pi(b - \sqrt{a^2 - x^2})^2 dx = \pi \int_{-a}^{a} 4b\sqrt{a^2 - x^2} dx$$

$$= 4\pi b \int_{-a}^{a} \sqrt{a^2 - x^2} dx = 2\pi^2 a^2 b$$

Shell Method：

$$x = \pm\sqrt{a^2 - (y-b)^2}, \; b - a \le y \le b + a$$

$$V = \int_{b-a}^{b+a} y \cdot 2\pi \cdot [\sqrt{a^2 - (y-b)^2} - (-\sqrt{a^2 - (y-b)^2})] \, dy$$

$$= 4\pi \int_{b-a}^{b+a} y\sqrt{a^2 - (y-b)^2} \; dy$$

$$= 4\pi \int_{b-a}^{b+a} (y-b)\sqrt{a^2 - (y-b)^2} + b\sqrt{a^2 - (y-b)^2} \; dy$$

$$= 4\pi \int_{-a}^{a} \underbrace{u\sqrt{a^2 - u^2}}_{odd\ function} + b\sqrt{a^2 - u^2} \; du \quad (Let\ u = y - b)$$

$$= 4\pi b \int_{-a}^{a} \sqrt{a^2 - u^2} \; du$$

$$= 2\pi^2 a^2 b$$

7. (10%) Let $y = f(x) = \sqrt{x}$ and $L(x)$ be the linearization of $f(x)$ at x=6. Find $L(x)$. Let $A = (36,\ f(36))$, $B = (37,\ f(36))$, $C = (37,\ f(37))$, and $D = (37,\ L(37))$. Let u be the area of the region bounded by $y = f(x)$, $\overline{AB}$, and $\overline{BC}$, v be the area of ΔABD. Determine the relation between u and v. By using the approximation $u \approx v$, one can obtain and estimate of the form $\sqrt{37} \approx \frac{1}{37}(K + \frac{1}{M})$, where K and M are integers. Find K and M.

Ans. L(x)＝______________. u____v (write <, =, >). K＝________. M＝_________.

....

Sol： $y = f(x) = \sqrt{x} \ \Rightarrow\ f'(x) = \dfrac{1}{2\sqrt{x}}$

$$L(x) = f(36) + f'(36)(x-36) = 6 + \frac{1}{12}(x-36)$$

$$L(37) = 6 + \frac{1}{12}$$

Since $f''(x) = -\frac{1}{4}x^{-3/2} < 0$ or $\begin{cases} f \ \textit{is concave down} \\ f' \ \textit{is decreasing} \end{cases}$ we know u<v.

$$u = \int_{36}^{37} \sqrt{x}dx - 6 = \frac{2}{3}x^{3/2}\Big|_{36}^{37} - 6 = \frac{2}{3}[(37)^{3/2} - 216] - 6 \approx v = 1 \times \frac{1}{12} \times \frac{1}{2}$$

$$(37)^{3/2} \approx 216 + \frac{3}{2}(\frac{1}{12} \times \frac{1}{2} + 6) = 225 + \frac{1}{16}$$

$$\sqrt{37} \approx \frac{1}{37}(225 + \frac{1}{16}) \ \Rightarrow\ K = 225,\ M = 16.$$