微甲統一教學二組95上期末考參考答案

1. (14%) 求極限 (limit) :

   (a) $\lim\limits_{x\to 0}\dfrac{2^{\sin x}-1}{e^x-1}$

   Solution:

   由 L'Hôpital Rule
   $\lim\limits_{x\to 0}\frac{2^{\sin x}-1}{e^x-1}=\lim\limits_{x\to 0}\frac{\ln 2\cdot 2^{\sin x}}{e^x}=\ln 2.$

   (b) $\lim\limits_{x\to 0}\dfrac{\int_0^{x^2}\frac{t}{\sqrt{1+t^3}}dt}{x^4}$

   Solution:

   $\lim\limits_{x\to 0}\frac{\int_0^{x^2}\frac{t}{\sqrt{1+t^3}}dt}{t^4}=\lim\limits_{x\to 0}\frac{\frac{x^2}{\sqrt{1+x^6}}\cdot 2x}{4x^3}=\lim\limits_{x\to 0}\frac{2}{4(x^4+1)}=\frac{1}{2}.$

2. (6%) 求 $y=x\sin^{-1}x+\sqrt{1-x^2}$ 之導函數 (derivative)。

   Solution:

   $\frac{dy}{dx}=\frac{x}{\sqrt{1-x^2}}+\sin^{-1}x+\frac{-x}{\sqrt{1-x^2}}=\sin^{-1}x.$

3. (40%) 求積分 (integral) :

   (a) $\displaystyle\int_1^e t(\ln t)^2dt$

   Solution:

   $$\begin{aligned}&\int_1^e t(\ln t)^2dt\\ &=\tfrac{1}{2}\int_1^e(\ln t)^2dt^2\\ &=\tfrac{1}{2}\{t^2(\ln t)^2|_1^e-2\int_1^e t(\ln t)dt\}\\ &=\tfrac{e^2}{2}-\tfrac{1}{2}\int_1^e\ln t d(t^2)\\ &=\tfrac{e^2}{2}-\tfrac{1}{2}\{t^2\ln t|_1^e-\int_1^e tdt\}\\ &=\tfrac{e^2}{2}-\tfrac{e^2}{2}+\tfrac{1}{2}(\tfrac{t^2}{2}|_1^e)\\ &=\tfrac{e^2}{4}-\tfrac{1}{4}.\end{aligned}$$

(b) $\displaystyle\int \frac{2x+1}{(x^2+1)^2}dx$

<u>Solution:</u>

$\int \frac{2x+1}{(x^2+1)^2}dx = \int \frac{2x}{(x^2+1)^2}dx + \int \frac{1}{(x^2+1)^2}dx = -\frac{1}{(x^2+1)} + \int \frac{1}{(x^2+1)^2}dx.$

$$\left\{ \begin{array}{l} \int \frac{1}{(x^2+1)^2}dx \overset{x=\tan\theta}{=} \int \frac{d\theta}{\sec^2\theta} = \int \cos^2\theta d\theta = \int \frac{1+\cos 2\theta}{2}d\theta \\ \qquad = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta = \frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{x^2+1} + C \end{array} \right\}$$

$= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{1}{x^2+1} - \frac{1}{x^2+1} + C.$

(c) $\displaystyle\int \frac{xdx}{\sqrt{8-2x^2-x^4}}$

<u>Solution:</u>

$\int \frac{xdx}{\sqrt{8-2x^2-x^4}}$

$\overset{u=x^2}{=} \frac{1}{2}\int \frac{du}{\sqrt{8-2u-u^2}}$

$= \frac{1}{2}\int \frac{du}{\sqrt{9-(u+1)^2}}$

$= \frac{1}{2}\int \frac{d(\frac{u+1}{3})}{\sqrt{1-(\frac{u+1}{3})^2}}$

$= \frac{1}{2}\sin^{-1}(\frac{u+1}{3}) + C$

$= \frac{1}{3}\sin^{-1}(\frac{x^2+1}{3}) + C.$

(d) $\displaystyle\int_0^\infty x^2 e^{-x}dx$

<u>Solution:</u>

利用 $\int x^n e^{-x}dx = -x^n e^{-x} + n\int x^{n-1}e^{-x}dx$

則 $\int_0^\infty x^2 e^{-x}dx = \lim\limits_{a\to\infty}\int_0^a x^2 e^{-x}dx$

$= \lim\limits_{a\to\infty}(-x^2e^{-x}|_0^a + 2\int_0^a xe^{-x}dx)$

$= 2\lim\limits_{a\to\infty}\int_0^a xe^{-x}dx$

$= 2\lim\limits_{a\to\infty}(-e^{-x}|_0^a + \int_0^a e^{-x}dx)$

$= 2\lim\limits_{a\to\infty}\int_0^a e^{-x}dx$

$= 2\lim\limits_{a\to\infty}(-e^{-x}|_0^a)$

$= 2\lim\limits_{a\to\infty}(-e^{-a}+1)$

$= 2.$

4. (10%)

   (a) 求兩曲線 $r = \cos\theta$ 及 $r = 1 - \cos\theta$ 之所有交點。

   Solution:
   $\cos\theta = 1 - \cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$.
   故交點 $(\frac{1}{2}, \pm\frac{\pi}{3})$。再由圖可知原點爲交點。

   (b) 求在圓 $r = \cos\theta$ 之內部, 且在心臟線 (cardioid) $r = 1 - \cos\theta$ 之外部的區域面積。

   Solution:
   $A = \int_0^{\frac{\pi}{3}} \cos^2\theta - (1 - \cos\theta)^2 d\theta.$
   $= \int_0^{\frac{\pi}{3}} (2\cos\theta - 1)d\theta = 2\sin\theta - \theta|_0^{\frac{\pi}{3}} = \sqrt{3} - \frac{\pi}{3}.$

5. (10%) 求曲線 $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ 之全長。

   Solution:
   $\begin{cases} x = \cos^3\theta \\ y = \sin^3\theta \end{cases}, \theta \in [0, 2\pi]$
   $\frac{dx}{d\theta} = -3\cos^2\theta\sin\theta.$
   $\frac{dy}{d\theta} = 3\sin^2\theta\cos\theta.$
   $ds = 3|\cos\theta\sin\theta|d\theta$
   $S = \int ds = 4\int_0^{\frac{\pi}{2}} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} d\theta$
   $= 12\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta d\theta = 6\int_0^{\frac{\pi}{2}} \sin 2\theta d\theta = -3\cos 2\theta|_0^{\frac{\pi}{2}} = 3 + 3 = 6$

6. (10%) 令 R 爲曲線 $y = 1 + \sin x$, $0 \le x \le 2\pi$, 之下方及 x-軸 上方的區域。將 R 繞 y-軸 旋轉所得之旋轉體體積爲何?

   Solution:
   $V = \int_0^{2\pi} 2\pi x(1 + \sin x)dx == 2\pi(\frac{\pi^2}{2} - x\cos x \sin x|_0^{2\pi}) = 4\pi^2(\pi - 1)$

7. (10%) 解微分方程 (differential equation) :

$$\begin{cases} \sec x \dfrac{dy}{dx} = e^{y+\sin x} & , x \in (-\frac{\pi}{2}, \frac{\pi}{6}). \\ \\ y(0) = 0 \end{cases}$$

   Solution:
   $\int \frac{dy}{e^y} = \int \cos x e^{\sin x} dx \Rightarrow -e^{-y} = e^{\sin x} + C.$
   因爲 $y(0) = 0$, 故 $-1 = C + 1 \Rightarrow C = -2$
   $\Rightarrow -e^{-y} = e^{\sin x} - 2 \Rightarrow e^{-y} = -e^{\sin x} + 2 \Rightarrow -y = ln(2 - e^{\sin x})$
   $\Rightarrow y = -ln(2 - e^{\sin x}).$