94學年度下學期微積分甲二組期中考

(14%) 1. Given curve $r = 1 + \cos\theta$.

(a) Find the length of the curve.

(b) Find the area of the region that is bounded by the curve.

Solution: (a)

$$
\begin{aligned}
\text{周長 } S &= 2\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\
&= 2\int_0^{\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2} d\theta \\
&= 2\int_0^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta \\
&= 4\int_0^{\pi} \sqrt{\frac{1+\cos\theta}{2}}\, d\theta \\
&= 4\int_0^{\pi} \left|\cos\frac{\theta}{2}\right|\, d\theta \\
&= 4\cdot 2\cdot \sin\frac{\theta}{2}\Big|_0^{\pi} \\
&= 8
\end{aligned}
$$

(b)

$$
\begin{aligned}
& 2\int_0^{\pi} \frac{1}{2} r^2\, d\theta \\
=& \int_0^{\pi} (1+\cos\theta)^2\, d\theta \\
=& \int_0^{\pi} (1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta)\, d\theta \\
=& \frac{3}{2}\pi
\end{aligned}
$$

(12%) 2. (a) Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2yx^3}{y^2+x^6}$.

(b) Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$.

Solution:

(a) Let $f(x,y) = \dfrac{2yx^3}{y^2+x^6}$.

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} 0 = 0$$

$$\lim_{x\to 0} f(x,x^3) = \lim_{x\to 0} \frac{2x^6}{x^6+x^6} = \lim_{x\to 0} 1 = 1$$

Since $0 \neq 1$, limit does not exist.

(b) Let $f(x,y)=\dfrac{xy}{\sqrt{x^2+y^2}}$, $x=r\cos\theta$, $y=r\sin\theta$.

$$\lim_{r\to 0} f(r\cos\theta, r\sin\theta)=\lim_{r\to 0}\frac{r^2\cos\theta\sin\theta}{r}=\lim_{r\to 0} r\cos\theta\sin\theta=0$$

So

$$\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}}=0.$$

(12%) 3. For $f(x,y,z)=\ln(x^2+y^2)+z$.

(a) Find ∇f.

(b) Consider the cylinder of radius 5 with axis along the z-axis. Find the normal vector to the cylinder at the point $(3,-4,4)$.

(c) Find the rate of change of f in the direction normal to the cylinder at the point $(3,-4,4)$.

Solution:

$\nabla f=<\dfrac{2x}{x^2+y^2},\dfrac{2y}{x^2+y^2},1>$.

The cylinder is $x^2+y^2=25$. So its normal vector is $<2x,2y,0>$ which equals $<6,-8,0>$ at $(3,-4,4)$. A unit vector in this direction is

$$\vec{u}=<\frac{3}{5},-\frac{4}{5},0>.$$

The rate of change of f in this direction is

$$\nabla f\cdot\vec{u}=<\frac{6}{5},-\frac{8}{5},1>\cdot<\frac{3}{5},-\frac{4}{5},0>=\frac{46}{15}.$$

(10%) 4. Find the parametric equation of the tangent line to the curve of intersection of the surfaces $x^2+2y^2+z^2=4$ and $x^2+y^2-z^2=1$ at the point $(1,1,1)$.

Solution:

$f=x^2+2y^2+z^2-4$, $g=x^2+y^2-z^2-1$, $P=(1,1,1)$.

$\nabla f=<2x,4y,2z>$, $\nabla g=<2x,2y,-2z>$.

$\nabla f(P)=<2,4,2>$, $\nabla g(P)=<2,2,-2>$.

$$\nabla f(P)\times\nabla g(P)=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&4&2\\2&2&-2\end{vmatrix}=<-12,8,-4>.$$

切線爲 $\begin{cases}x=1-12t\\y=1+8t\\z=1-4t\end{cases}$ 或 $\begin{cases}x=1+3t\\y=1-2t\\z=1+t\end{cases}$.

(18%) 5. (a) Suppose $f_x(0,0)=2$, $f_y(0,0)=3$. Suppose $v=g(u)$ satisfies $f(u,v^2+v)=0$ and $g(0)=0$. Find $g'(0)$.

(b) Find all points at which the direction of fastest increase of the function $f(x,y)=x^2+y^2-2x-4y$ is $2\mathbf{i}+\mathbf{j}$.

Solution:

(a) Let $x = u$, $y = v^2 + v$.

$$\frac{df}{du} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\frac{dv}{du}.$$

$f(u, v^2 + v) = 0$

$f_x(u, v^2 + v) \cdot 1 + f_y(u, v^2 + v) \cdot (2v + 1) \cdot g'(u) = 0.$

$u = 0, v = 0$ 代入

$f_x(0, 0) + f_y(0, 0) \cdot 1 \cdot g'(0) = 0.$

$2 + 3 \cdot g'(0) = 0.$

$g'(0) = \frac{-2}{3}.$

(b) $\nabla f =< 2x - 2, 2y - 4 >= k < 2, 1 >$, $k > 0$.
$2x - 2 = 2(2y - 4)$
$\Rightarrow x - 2y + 3 = 0$, $x > 1$.

(12%) 6. Find the absolute minimum and maximum values of $f(x, y) = xy^2$ on the curve $x^2 + 7xy + y^2 = 45$, $x \geq 0$, $y \geq 0$.

Solution:

Let $g = x^2 + 7xy + y^2 - 45$.
$\nabla g =< 2x + 7y, 7x + 2y >$
$\nabla f =< y^2, 2xy >$
$$\begin{cases} y^2 = \lambda(2x + 7y) - -(1) \\ 2xy = \lambda(7x + 2y) - -(2) \end{cases}$$
$(1) \times (7x + 2y) - (2) \times (2x + 7y)$
$y(2y^2 - 7xy - 4x^2) = 0$
$\Rightarrow y = 0$, $y = 4x$, $y = -\frac{x}{2}$.
$y = -\frac{x}{2}$ 不合.
代入 g
$y = 0 \Rightarrow x = 3\sqrt{5}$.
$y = 4x \Rightarrow x = 1$, $y = 4$.
$f(3\sqrt{5}, 0) = 0$ 最小值.
$f(1, 4) = 16$ 最大值.

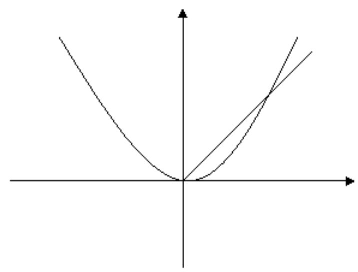
(10%) 7. Find the volume of the solid in the 1st octant bounded by the surface $z = 9 - y^2$ and the plane $x = 2$.

Solution:

$$\int_0^2 \int_0^3 (9 - y^2)\, dy\, dx = \int_0^2 9y - \frac{y^3}{3}\Big|_0^3\, dx = \int_0^2 18\, dx = 36.$$

(12%) 8. Evaluate $\displaystyle\int_0^1 \int_y^{\sqrt{y}} \frac{\sin x}{x}\, dx\, dy.$

Solution:



$$\begin{aligned}
\int_0^1 \int_y^{\sqrt{y}} \frac{\sin x}{x}\, dx\, dy &= \int_0^1 \int_{x^2}^{x} \frac{\sin x}{x}\, dy\, dx = \int_0^1 (x - x^2)\frac{\sin x}{x}\, dx \\
&= \int_0^1 \sin x - x \sin x\, dx = [-\cos x + x\cos x - \sin x]_0^1 \\
&= -\cos 1 + \cos 1 - \sin 1 - (-1) = 1 - \sin 1
\end{aligned}$$

where $\displaystyle\int x \sin x\, dx = -x\cos x - \int -\cos x\, dx = -x\cos x + \sin x + C.$