94學年上學期微甲二組期中考

(7分)1. 求 $\lim_{x\to 0}\frac{|2x-1|-|2x+1|}{x}$

Solution:

$\lim_{x\to 0^+}\frac{|2x-1|-|2x+1|}{x}=\lim_{x\to 0^+}\frac{1-2x-(2x+1)}{x}=\lim_{x\to 0^+}\frac{-4x}{x}=-4$
$\lim_{x\to 0^-}\frac{|2x-1|-|2x+1|}{x}=\lim_{x\to 0^-}\frac{1-2x-(2x+1)}{x}=\lim_{x\to 0^-}\frac{-4x}{x}=-4$
$\Rightarrow \lim_{x\to 0}\frac{|2x-1|-|2x+1|}{x}=-4$

(7分)2. 求 $f(x)=(1+3x)^{\frac{1}{3}}$ 在 $x=0$ 的線性逼近式.

Solution:

$$\begin{aligned} f(x) &\simeq f(0)+f'(x)|_{x=0}\ x \\ &= 1+\frac{1}{3}(1+3x)^{\frac{1}{3}-1}\cdot 3|_{x=0}\ x \\ &= 1+x. \end{aligned}$$

(12分)3. 給定曲線 $xy^2+x+y=3$, 求 y', y'' 在點 $(1,1)$的值.

Solution:

$y^2+2xyy'+1+y'=0,\ y'=\frac{-1-y^2}{1+2x}$
$y'_{(1,1)}=-\frac{2}{3}$
$2yy'+2yy'+2xy'^2+2xyy''+y''=0,\ y''=\frac{-4yy'-2xy'^2}{1+2xy}$
$y''_{(1,1)}=\frac{16}{27}$

(12分)4. $y=1-x^2$ 在第一象限之切線與座標軸圍成之三角形面積, 最小為多少?

Solution:

設切點 $(a,1-a^2)$ 切線方程式為 $y=-2ax+a^2+1$. x, y 截距為 $\frac{a^2+1}{2a}$, a^2+1, 所以面積為 $\frac{(a^2+1)^2}{4a}=A(a)$. $A'(a)=0$ 得 $a=\frac{1}{\sqrt{3}}$, $A=\frac{4\sqrt{3}}{9}$ min.

(12分)5. 令 $f(x)=x^{\frac{1}{3}}(x-4)$

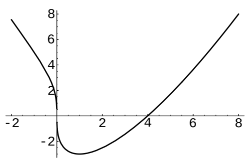
(a) 指出其遞增、遞減的區間.
(b) 指出其凹向上、凹向下的區間.
(c) 繪圖並標明極值和反曲點.

Solution:

$y' = \dfrac{4}{3}\dfrac{x-1}{x^{\frac{2}{3}}} \qquad x=1 \Rightarrow y'(0)=0, \quad x=0 \ \ y'$不存在

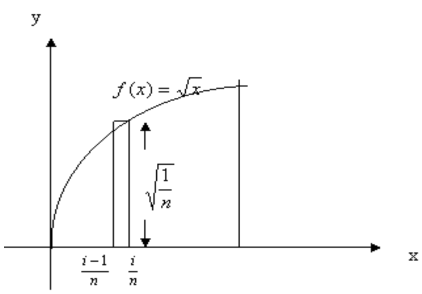
$y'' = \dfrac{4}{9}\dfrac{x+2}{x^{\frac{5}{3}}} \qquad x=-2 \Rightarrow y''=0, \quad x=0 \ \ y''$不存在

x		-2		0		1	
y		$6\sqrt[3]{2}$		0		-3	
y'		$-$		$\times$	$-$	0	$+$
y''	$+$	0	$-$	$\times$		$+$	
		反曲點		反曲點		極小	



(12分)6. 求 $\lim\limits_{n\to\infty}\left(\sqrt{\dfrac{1}{n^3}}+\sqrt{\dfrac{2}{n^3}}+\cdots+\sqrt{\dfrac{n}{n^3}}\right)$.

Solution:



$$\begin{aligned}
& \lim_{n\to\infty}\left(\sqrt{\frac{1}{n^3}}+\sqrt{\frac{2}{n^3}}+\cdots+\sqrt{\frac{n}{n^3}}\right) \\
= & \lim_{n\to\infty}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\cdots+\sqrt{\frac{n}{n}}\right)\frac{1}{n} \\
= & \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\frac{i}{n}}\cdot\frac{1}{n} \\
= & \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n} \\
= & \int_0^1 \sqrt{x}\ dx \\
= & \left[\frac{2}{3}x^{3/2}\right]_0^1 \\
= & \frac{2}{3}-0 \\
= & \frac{2}{3}
\end{aligned}$$

7. 求下列積分

(7分) (a) $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}}\ dx$

(7分) (b) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2\theta\tan^2\theta\ d\theta.$

Solution:

(a) $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}}\ dx = \frac{8}{3}$

(b) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2\theta\tan^2\theta\ d\theta = \sqrt{3} - \frac{1}{3}$

(12分)8. 設 $f(\theta) = \displaystyle\int_0^{\cos\theta} \sin\left(t^2\right)\ dt$, 求 $f'(\theta)$, $f''(\theta)$.

Solution:

$f'(\theta) = \sin\left(\cos^2\theta\right)\cdot(-\sin\theta)$
$f''(\theta) = \cos\left(\cos^2\theta\right)\cdot 2\cos\theta\,(-\sin\theta)^2 - \sin\left(\cos^2\theta\right)\cos\theta$

(12分)9. 計算 $y = 6x^2$, $x = 1$, $y = 0$ 所圍成的區域對 $x = -1$ 旋轉所得的旋轉體體積.

Solution:

方法一:

$$\int_0^1 2\pi\,(x+1)\,6x^2\ dx = 2\pi\int_0^1 6x^3+6x^2\ dx = 2\pi\left[\frac{3x^4}{2}+2x^3\right]_0^1 = 2\pi\left(\frac{3}{2}+2\right) = 7\pi$$

方法二:

$$\begin{aligned}\int_0^6 \pi\left[2^2 - \left(1+\sqrt{\frac{y}{6}}\right)^2\right]\ dy &= \pi\int_0^6\left[3 - \frac{2}{\sqrt{6}}\sqrt{y} - \frac{y}{6}\right]\ dy \\ &= \pi\left(18 - \frac{2}{\sqrt{6}}\cdot\frac{2}{3}\cdot 6^{\frac{3}{2}} - \frac{1}{6}\cdot\frac{36}{2}\right) = 7\pi\end{aligned}$$