93學年下學期微甲一組期中考

1. Find the arc length function $s(t)$, unit tangent vector $T(t)$, unit normal vector $N(t)$, and the curvature $\kappa(t)$ of the curve

$$R(t) = \left(e^t, e^{-t}, \sqrt{2}t\right),\ t \geq 0,$$

with initial point $t = 0$.

Solution:

$S(t) = e^t - e^{-t}$

$\overrightarrow{T}(t) = \dfrac{1}{e^t + e^{-t}} \left\langle e^t, -e^{-t}, \sqrt{2} \right\rangle$ or $\dfrac{1}{e^{2t}+1} \left\langle e^{2t}, -1, \sqrt{2}e^t \right\rangle$

$\overrightarrow{N}(t) = \dfrac{1}{e^t + e^{-t}} \left\langle \sqrt{2}, \sqrt{2}, e^{-t} - e^t \right\rangle$ or $\dfrac{1}{e^{2t}+1} \left\langle \sqrt{2}e^t, \sqrt{2}e^t, 1 - e^{2t} \right\rangle$

$\kappa(t) = \dfrac{\sqrt{2}}{(e^t + e^{-t})^2}$ or $\dfrac{\sqrt{2}e^{2t}}{(e^{2t}+1)^2}$.

2. Let $z = f(x, y)$ be a function with continuous second-order partial derivatives, and $x = u^2 \cos v$, $y = u^2 \sin v$. Express the Laplacian $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}$ in terms of u, v, $\dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}, \dfrac{\partial^2 z}{\partial u^2}, \dfrac{\partial^2 z}{\partial v^2}$.

Solution:

$\dfrac{\partial z}{\partial u} = 2u \cos v f_x + 2u \sin v f_y$

$\dfrac{\partial z}{\partial v} = -u^2 \sin v f_x + u^2 \cos v f_y$

$\dfrac{\partial^2 z}{\partial u^2} = 4u^2 \cos^2 v f_{xx} + 4u^2 \sin^2 v f_{yy} + 8u^2 \sin v \cos v f_{xy} + 2 \cos v f_x + 2 \sin v f_y$

$\dfrac{\partial^2 z}{\partial v^2} = u^4 \sin^2 v f_{xx} + u^4 \cos^2 v f_{yy} - 2u^4 \sin v \cos v f_{xy} - u^2 \cos v f_x - u^2 \sin v f_y$

$$\frac{\partial^2 z}{\partial u \partial v} = -2u^3 \sin v \cos v f_{xx} + 2u^3 \sin v \cos v f_{yy} + 2u^3 \left(\cos^2 v - \sin^2 v\right) f_{xy} - 2u \sin v f_x + 2u \cos v f_y$$

$$\begin{aligned}
\frac{1}{u^4}\frac{\partial^2 z}{\partial v^2} + \frac{1}{4u^2}\frac{\partial^2 z}{\partial u^2} &= f_{xx} + f_{yy} - \frac{1}{u^2}\cos v f_x - \frac{1}{u^2}\sin v f_y + \frac{1}{2}\frac{1}{u^2}\cos v f_x + \frac{1}{2}\frac{1}{u^2}\sin v f_y \\
&= f_{xx} + f_{yy} - \frac{1}{4u^3}\frac{\partial z}{\partial u}
\end{aligned}$$

$$\Rightarrow \frac{1}{u^4}\frac{\partial^2 z}{\partial v^2} + \frac{1}{4u^2}\frac{\partial^2 z}{\partial u^2} + \frac{1}{4u^3}\frac{\partial z}{\partial u} = f_{xx} + f_{yy}$$

3. Suppose $f(x, y)$ is differentiable at (a, b), $u = < 1, 0 >$, $v = \left\langle \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} \right\rangle$, $D_u f(a, b) = 3$, and $D_v f(a, b) = \sqrt{2}$.

(a) Find $\nabla f(a,b)$.

(b) What is the maximum of $D_w f(a,b)$ for any w.

(c) Find all unit vectors $w = \langle w_1, w_2 \rangle$ such that $D_w f(a,b) = 0$.

Solution:

(a) $\nabla f(a,b) = (3,-1) = 3\mathbf{i} + (-1)\mathbf{j}$

(b) 當 $\overrightarrow{w} = \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$ 時,
$D_{\overrightarrow{w}} f(a,b) = \sqrt{10}$, 有最大值。

(c) 當 $\overrightarrow{w} = \pm\left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$ 時,
$D_w f(a,b) = 0$

4. Given $f(x,y) = 3xy - 2xy^2 - x^2y$.

(a) Find all critical points, and determine its nature.

(b) Use the method of Lagrange multipliers to find the maximum and minimum of $f(x,y)$ on the plane region: $xy \geq \frac{1}{8}$, $0 \leq x \leq 1$ and $0 \leq y \leq 1$ (You must use the indicated method in order to receive any partial credits.)

5. Consider the iterated integral

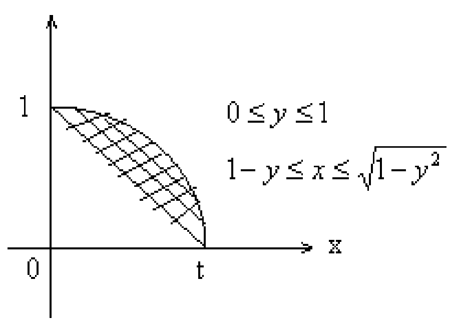
$$\int_0^1 \left(\int_{1-y}^{\sqrt{1-y^2}} \frac{1}{\sqrt{x^2+y^2}}\, dx \right) dy$$

(a) Sketch the region of integration.

(b) Evaluate the integral by reversing the order of integration.

Solution:

(a)



(b)

$$\begin{aligned}
\int_0^1 \int_{1-y}^{\sqrt{1-y^2}} \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy &= \int_0^{\frac{\pi}{2}} \int_{\frac{1}{\cos\theta+\sin\theta}}^{1} \frac{1}{r} r\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{\cos\theta+\sin\theta}\right) d\theta \\
&= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta+\sin\theta}\, d\theta \\
&= \frac{\pi}{2} - \sqrt{2}\ln\left(\sqrt{2}+1\right)
\end{aligned}$$