92學年下學期微甲二組期末考

1. (10分)求 $\iint\limits_D e^{x^2-xy+y^2}\ dA$, 此處 $D=\{(x,y)|x^2-xy+y^2\le a^2\}$。

   (提示:$x=u-\frac{1}{3}v,\ y=u+\frac{1}{3}v$)

2. (10分)求 $\iiint\limits_H z^3\sqrt{x^2+y^2+z^2}\ dV$, 此處 H 為一球心在原點, 半徑為 1, 在 xy 平面上方的實心半球體區域。

   Solution:

   Use spherical word

$$I=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}d\phi\int_0^1\left(\rho^3\cos^3\phi\right)\left(\rho^2\sin\rho\right)d\rho=\frac{\pi}{14}$$

3. (10分)設 E 為由 $z=(x^2+y^2)^{\frac{1}{2}}$, $x^2+y^2=9$, 與 $z=0$ 三曲面所圍成的區域, 該區域的密度函數為 $\rho(x,y,z)=z$。試求 E 的質量與質心。

   Solution:

   Let $D=\{(x,y)|x^2+y^2<\rho\}$, then

$$\begin{aligned} m &= \iiint\limits_E \rho(x,y,z)\ dV=\iiint\limits_E z\ dV=\iint\limits_D\left(\int_0^{\sqrt{x^2+y^2}} z\ dz\right)dA \\ &= \iint\limits_D\left(\frac{1}{2}z^2\Big|_0^{\sqrt{x^2+y^2}}\right)\ dA=\frac{1}{2}\iint\limits_D x^2+y^2\ dA=\frac{1}{2}\int_0^{2\pi}\int_0^3 r^2\cdot r\ drd\theta \\ &= \frac{1}{2}\int_0^{2\pi}1\ d\theta\cdot\int_0^3 r^3\ dr=\pi\cdot\frac{1}{4}\cdot r^4\Big|_0^3=\frac{81}{4}\pi. \end{aligned}$$

   Because of the symmetry of E and ρ about the yz-plane and xz-plane, we get $M_{xz}=M_{yz}=0$, therefore $\overline{x}=\overline{y}=0$. The moment

$$\begin{aligned} M_{xy} &= \iiint\limits_E z\cdot\rho(x,y,z)\ dV=\iint\limits_D\left(\int_0^{\sqrt{x^2+y^2}} z^2\ dz\right)\ dA \\ &= \frac{1}{3}\iint\limits_D\left(x^2+y^2\right)^{\frac{3}{2}}\ dA=\frac{1}{3}\int_0^{2\pi}\int_0^3 r^3\cdot r\ drd\theta \\ &= \frac{1}{3}\int_0^{2\pi}1\ d\theta\cdot\int_0^3 r^4\ d\theta=\frac{3}{5}\cdot 2\pi=\frac{162}{5}\pi \end{aligned}$$

   $\therefore$ the center of mass is

$$(\overline{x},\overline{y},\overline{z})=(\frac{M_{yz}}{m},\frac{M_{xz}}{m},\frac{M_{xy}}{m})=(0,0,\frac{8}{5})$$

4. (10分)在力場 $\overrightarrow{F} = x^2y\overrightarrow{i} - xy^2\overrightarrow{j}$ 中, 求將一粒子以逆時針方向沿著圓 $x^2+y^2=4$ 移動一圈所做的功。

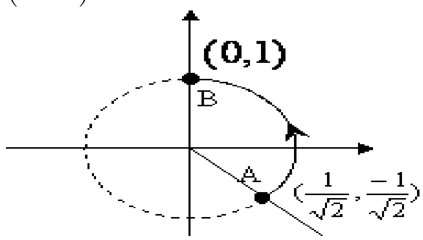
   Solution:

   Let C denote the counterclockwise circle $x^2+y^2=4$ and D denot the region $x^2+y^2 \le 4$. Then the work

$$W = \int_C \overrightarrow{F}\cdot d\overrightarrow{r} = \int_C x^2y\,dx - xy^2\,dy = \iint_D -\left(y^2+x^2\right)\,dA = \int_0^{2\pi}\int_0^2 -r^3\,drd\theta$$

   Thus $W = -8\pi$.

5. (10分)給定圓 C 上由點 A 到點 B 的有向曲線, 相關圖示如下:



   求沿 C 由點 A 至點 B 的線積分 $\int_{C[A,B]} x\,ds$ 之值。

   Solution: 令右半圓周上由 A 到 B 的弧參數化, 設$x = a\cos t$, $y = a\sin t$, $-\frac{\pi}{4} \le t \le \frac{\pi}{2}$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt = a\,dt$$

$$\int_{\widehat{AB}} x\,ds = \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} (a\cos t)(adt) = a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \cos t\,dt = a^2[\sin t]_{-\frac{\pi}{4}}^{\frac{\pi}{2}} = a^2\left(1+\frac{1}{\sqrt{2}}\right)$$

6. (10分)若向量場為 $\overrightarrow{F}(x,y,z) = z\overrightarrow{i} + y\overrightarrow{j} - x\overrightarrow{k}$, 有向曲線 $C: \overrightarrow{\gamma}(t) = t\overrightarrow{i} + \sin t\overrightarrow{j} + \cos t\overrightarrow{k}$, $0 \le t \le \pi$, 試求 $\int_C \overrightarrow{F}\cdot d\overrightarrow{r}$。

7. (10分)設向量場 $\overrightarrow{F}(x,y) = (3+2xy)\overrightarrow{i} + (x^2-3y^2)\overrightarrow{j}$

   (a) 求位能函數 (potential function) $f(x,y)$ 使得 $\nabla f = \overrightarrow{F}$。

   (b) 求 $\int_C F\cdot dr$ 之值, 此處有線段 $C: \overrightarrow{\gamma}(t) = \cos t\overrightarrow{i} + \sin t\overrightarrow{j}$, $0 \le t \le \frac{\pi}{2}$。

   Solution:

   (a) $f_x = 3+2xy,\ f_y = x^2-3y^2 \Rightarrow f(x,y) = 3x + x^2y - y^3 + k.$

   (b) $\int_C F\cdot dr = f(0,1) - f(1,0) = -4.$

8. (10分)設向量場 $\overrightarrow{F}(x,y,z) = xy\overrightarrow{i} + yz\overrightarrow{j} + zx\overrightarrow{k}$, C 為以 $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ 為頂點之三角形曲線, 試求 $\oint_C \overrightarrow{F}\cdot\overrightarrow{T}\,ds$。

   Solution:

$S: x+y+z=1,\ \overrightarrow{n}=\frac{1}{\sqrt{3}}<1,1,1>,\ \text{curl}F=-<y,z,x>$

$\oint_C F\cdot T\ ds=\iint_S \text{curl}F\cdot\overrightarrow{n}\ dS=\frac{1}{2}.$

9. (10分)若向量場 $\overrightarrow{F}=x^2yz\overrightarrow{i}+yz^2\overrightarrow{j}+z^3e^{xy}\overrightarrow{k}$

   (a) 求 $\overrightarrow{F}$ 的旋度 (curl)。

   (b) 計算 $\iint_S \text{curl}\overrightarrow{F}\cdot\overrightarrow{n}\ dS$, 此處曲面 S 為由球面 $x^2+y^2+z^2=5$ 被 $z=1$ 所截出的上方部分, 而 $\overrightarrow{n}$ 是球面上的往外單位向量。

Solution:

(a)

$$\begin{aligned}\text{curl}\overrightarrow{F} &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & yz^2 & z^3e^{xy} \end{vmatrix} \\ &= \left(xz^3e^{xy}-2yz\right)\overrightarrow{i}-\left(xz^3e^{xy}-x^2y\right)\overrightarrow{j}-x^2z\overrightarrow{k}.\end{aligned}$$

(b) By Stokes's theorem

$$\begin{aligned}&\iint_S \text{curl}\overrightarrow{F}\cdot\overrightarrow{n}\ dS \\ =& \oint_C \overrightarrow{F}\cdot d\overrightarrow{r} \\ =& \oint_C x^2yz\ dx+yz^2\ dy+z^3e^{xy}\ dz \\ =& \int_0^{2\pi} 4\cos t\sin t(-2\sin t)+2\sin t(2\cos t)\ dt \\ =& \int_0^{2\pi} -8\sin^2 t\cos t+2\sin 2t\ dt \\ =& \left[\frac{-8\sin^3 t}{3}-\cos 2t\right]_0^{2\pi}=0.\end{aligned}$$

10. (10分) 求 $\iint_S \overrightarrow{F}\cdot\overrightarrow{n}\ dS$, 其中 $\overrightarrow{F}=x^2\overrightarrow{i}+xy\overrightarrow{j}+z\overrightarrow{k}$, 曲面 S 是由拋物面 $z=x^2+y^2$ 與平面 $z=1$ 所圍成實體的表面。

Solution:

$$
\begin{aligned}
\iint\limits_S \overrightarrow{F} \cdot \overrightarrow{n}\ dS &= \iiint\limits_V \operatorname{div} \overrightarrow{F}\, dV \\
&= \iiint\limits_V 2x + y + 1\ dV \\
&= \iint\limits_D \left(\int_{x^2+y^2}^{1} 2x + y + 1\ dz \right)\ dA, \quad \text{where } D = \{(x,y) : x^2 + y^2 \leq 1\} \\
&= \iint\limits_D (1 - x^2 - y^2)(2x + y + 1)\ dA \\
&= \int_0^{2\pi} \int_0^1 \left(1 - r^2\right) (2r\cos\theta + r\sin\theta + 1) r\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^1 r - r^3 + (2\cos\theta + \sin\theta)(r^2 - r^4)\ dr\ d\theta \\
&= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4} + (2\cos\theta + \sin\theta) \left(\frac{r^3}{3} - \frac{r^5}{5} \right) \right]_0^1\ d\theta \\
&= \int_0^{2\pi} \frac{1}{4} + \frac{4}{15}\cos\theta + \frac{2}{15}\sin\theta\ d\theta \\
&= \left[\frac{1}{4}\theta + \frac{4}{15}\sin\theta - \frac{2}{15}\cos\theta \right]_0^{2\pi} \\
&= \frac{1}{4} \cdot 2\pi = \frac{\pi}{2}.
\end{aligned}
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