微積分 (甲) 學期考試

2003年1月11日.

1. 令R表示由 $y = 1 + \sin x$ 與 $y - 1$ 兩曲線在 $x = 0$ 與 $x = 2\pi$ 之間所圍區域, 試求由R繞y軸旋轉所得立體之體積. (12%)

解: $V = 2\pi\{\int_0^{\pi} x[(1+\sin x)-1]dx + \int_{\pi}^{2\pi} x[1-(1+\sin x)]dx\} = 2\pi(\int_0^{\pi} x\sin x dx - \int_{\pi}^{2\pi} x \sin x dx)$.
Because $\int x\sin x dx = \int x d(-\cos x) = -x\cos x + \int cos x dx = -x\cos x + \sin x + C$.
Hence, $\int_0^{\pi} x\sin x dx = -x\cos x + \sin x|_0^{\pi} = \pi$, $\int_{\pi}^{2\pi} x\sin x dx = -x\cos x + \sin x|_{\pi}^{2\pi} = -2\pi - \pi = -3\pi$.
Therefore, $V = 2\pi[\pi - (-3\pi)] = 8\pi^2$

2. 試求擺線 $x = t - \sin t$, $y = 1 - \cos t$ 一拱 $(0 \le t \le 2\pi)$ 之長. (12%)

解: Because $x = t - \sin t$, $y = 1 - \cos t$. Hence $\frac{dx}{dt} = 1 - \cos t$, $\frac{dy}{dt} = \sin t$

$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{(1-cost)^2 + \sin^2 t} dt \\ &= \int_0^{2\pi} \sqrt{2 - 2\cos t} dt \\ &= \int_0^{2\pi} \sqrt{4\sin^2 \frac{t}{2}} dt \\ &= 2\int_0^{2\pi} \sin\frac{t}{2} dt \\ &= 4(-\cos\frac{t}{2})|_0^{2\pi} \\ &= 4[1-(-1)] = 8 \end{aligned}$$

3. 試求不定積分 $\int \frac{e^{3x}}{e^{3x}+2e^{2x}+2e^{+1}}dx$. (12%)

解: 令 $u = e^x$, 則 $x = \ln u$, $dx = \frac{du}{u}$.

$$\begin{aligned} & \int \frac{e^{3x}}{e^{3x}+2e^{2x}+2e^{x}+1}dx \\ = & \int \frac{u^3}{u^3+2u^2+2u+1} \cdot \frac{du}{u} \\ = & \int \frac{u^2}{(u+1)(u^2+u+1)} du \end{aligned}$$

設 $\frac{u^2}{(u+1)(u^2+u+1)} = \frac{A}{u+1} + \frac{Bu+C}{u^2+u+1}$, 則 $u^2 = A(u^2+u+1) + (Bu+C)(u+1)$. 令 $u = -1$, 得 $A = 1$, 故 $-u-1 = (Bu+C)(u+1)$, 即 $Bu + C = -1$. 因此,

$$\text{原式} = \int \frac{du}{u+1} - \int \frac{du}{u^2+u+1}$$

$$
\begin{aligned}
&= \ln|u+1| - \int \frac{du}{(u+\frac{1}{2})^2+\frac{3}{4}} \\
&= \ln|u+1| - \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}(\frac{u+\frac{1}{2}}{\frac{\sqrt{3}}{2}}) + C \\
&= \ln(e^x+1) - \frac{2}{\sqrt{3}}\tan^{-1}(\frac{2e^x+1}{\sqrt{3}}) + C
\end{aligned}
$$

4. 試求極限 $\lim_{x\to 0} \frac{\cos 3x}{\cos x}^{\frac{1}{x^2}}$ 之值. (12%)

解:

$$
\begin{aligned}
& \lim_{x\to 0}(\frac{\cos 3x}{\cos x})^{\frac{1}{x^2}} \\
= \ & \lim_{x\to 0}\exp(\frac{\ln|\cos 3x| - \ln|\cos x|}{x^2}) \\
= \ & \exp(\lim_{x\to 0}\frac{\frac{-3\sin 3x}{\cos 3x} + \frac{\sin x}{\cos x}}{2x}) \\
= \ & \exp(\lim_{x\to 0}\frac{-3\tan 3x + \tan x}{2x}) \\
= \ & \exp(\lim_{x\to 0}\frac{-9\sec^2 3x + \sec^2 x}{2}) \text{或} = \exp[\lim_{x\to 0}(\frac{\sin 3x}{3x}\cdot\frac{-9}{2\cos 3x} + \frac{\sin x}{x}\cdot\frac{1}{2\cos x})] \\
= \ & \exp(-\frac{9}{2}+\frac{1}{2}) = \exp(-4) = e^{-4}.
\end{aligned}
$$

5. 設$p > 1$, 試求瑕積分 $\int_1^\infty \frac{(\ln x)^2}{x^p}dx$ 之值. (12%)

解:

$$
\begin{aligned}
& \int \frac{(\ln x)^2}{x^p}dx = \int x^{-p}(\ln x)^2 dx \\
= \ & \frac{1}{1-p}\int (\ln x)^2 dx^{1-p} \\
= \ & \frac{1}{1-p}[x^{1-p}(\ln x)^2 - \int x^{1-p} d(\ln x)^2] \\
= \ & \frac{1}{1-p}[x^{1-p}(\ln x)^2 - 2\int x^{-p}\ln x dx] \\
= \ & \frac{(\ln x)^2}{(1-p)x^{p-1}} - \frac{2}{(1-p)^2}\int \ln x dx^{1-p} \\
= \ & \frac{(\ln x)^2}{(1-p)x^{p-1}} - \frac{2}{(1-p)^2}[x^{1-p}\ln x - \int x^{1-p} d(\ln x)] \\
= \ & \frac{(\ln x)^2}{(1-p)x^{p-1}} - \frac{2\ln x}{(1-p)^2 x^{p-1}} + \frac{2}{(1-p)^2}\int x^{-p} dx \\
= \ & \frac{(\ln x)^2}{(1-p)x^{p-1}} - \frac{2\ln x}{(1-p)^2 x^{p-1}} + \frac{2}{(1-p)^3 x^{p-1}} + C
\end{aligned}
$$

因爲 $p>1$, $\lim_{x\to\infty}\frac{(\ln x)^2}{x^{p-1}}=\lim_{x\to\infty}\frac{\ln x}{2(p-1)x^{p-1}}=\lim_{x\to\infty}\frac{1}{2(p-1)^2x^{p-1}}=0$.
所以 $\int_1^\infty\frac{(\ln x)^2}{x^p}dx=\frac{(\ln x)^2}{(1-p)x^{p-1}}-\frac{2\ln x}{(1-p)^2x^{p-1}}+\frac{2}{(1-p)^3x^{p-1}}\Big|_1^\infty=-\frac{1}{(1-p)^3}=\frac{1}{(p-1)^3}$
依 L'Hôpital 法則 ($\frac{\infty}{\infty}$形) 或 $x^\alpha>>(\ln x)^\beta$ (as $x\to\infty$), $\alpha,\beta>0$ 皆可求得.

6. 設a,b爲常數, 且 $\lim_{x\to 0}(x^{-3}\sin 3x+ax^{-2}+b)=0$, 試求$a,b$ 之値. (12%)

解: $x^{-3}\sin 3x+ax^{-2}+b=\frac{\sin 3x+ax+bx^3}{x^3}$.
令 $f(x)=\sin 3x+ax+bx^3$, $g(x)=x^3$,
則 $\frac{f'(x)}{g'(x)}=\frac{3\cos 3x+a+3bx^2}{3x^2}$
$\frac{f''(x)}{g''(x)}=\frac{-9\sin 3x+6bx}{6x}=-\frac{9}{2}\cdot\frac{\sin 3x}{3x}+b\to-\frac{9}{2}+b$, 當 $x\to 0$.
依 L'Hôpital 法則, $\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{f''(x)}{g''(x)}=-\frac{9}{2}+b$,
故得 $-\frac{9}{2}+b=0$, 即 $b=\frac{9}{2}$.
又因 $\lim_{x\to 0}\frac{f'(x)}{g'(x)}=0$, 且 $\lim_{x\to 0}g'(x)=0$, 故 $\lim_{x\to 0}f'(x)=0$, 而 $f'(x)=3\cos 3x+a+3b^2\to 3+a$, 當 $x\to 0$, 故得 $3+a=0$, 即 $a=-3$.

7.

(1) 試寫出 $\tan^{-1}x$ 之 MacLaurin 級數及其收斂範圍. (4%)
(2) 試求級數 $\sum_{n=0}^\infty\frac{(-1)^n}{3^n(2n+1)}$ 之和. (8%)

解:

(1) $\tan^{-x}=\sum_{n=0}^\infty\frac{(-1)^n}{2n+1}x^{2n+1}$, $-1\le x\le 1$.
(2) 令 $x=\frac{1}{\sqrt{3}}$, 則

$$\tan^{-1}\frac{1}{\sqrt{3}}=\sum_{n=0}^\infty\frac{(-1)^n}{2n+1}\cdot\frac{1}{3^n}\cdot\frac{1}{\sqrt{3}},$$

故得

$$\sum_{n=0}^\infty\frac{(-1)^n}{3^n(2n+1)}=\sqrt{3}\tan^{-1}\frac{1}{\sqrt{3}}=\sqrt{3}\cdot\frac{\pi}{3}=\frac{\pi\sqrt{3}}{6}.$$

8. 設 $f(x)=x\sqrt{\frac{1-x}{1+x}}$, $-1<x<1$.

(a) 試求 $f(x)$ 之 MacLaurin 級數. (10%)
(b) 試求 $f^{(n)}(0)$. (6%)

解:

(a)

$$
\begin{aligned}
f(x) &= x\sqrt{\frac{1-x}{1+x}} = \frac{x(1-x)}{\sqrt{1-x^2}} = (x-x^2)(1-x^2)^{-\frac{1}{2}} \\
&= (x-x^2)[1+(-\frac{1}{2})(-x^2)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{1\cdot 2}(-x^2)^2+\cdots+ \\
&\quad \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(\frac{2n-1}{2})}{n!}(-x^{2n})+\cdots] \\
&= (x-x^2)(1-\frac{x^2}{2}+\frac{1\cdot 3}{2^2\cdot 2!}x^4+\cdots+\frac{1\cdot 3\cdots(2n-1)}{2^n(n!)}x^{2n}+\cdots) \\
&= x+\frac{x^3}{2}+\frac{1\cdot 3}{2^2\cdot 2!}x^5+\cdots+\frac{1\cdot 3\cdots(2n-1)}{2^n(n!)}x^{2n+1}+\cdots \\
&\quad -(x^2-\frac{x^4}{2}+\frac{1\cdot 3}{2^2\cdot 2!}x^6+\cdots+\frac{1\cdot 3\cdots(2n-1)}{2^n(n!)}x^{2n+2}+\cdots) \\
&= x-x^2+\frac{x^3}{2}-\frac{x^4}{2}+\cdots+\frac{1\cdot 3\cdots(2n-1)}{2^n(n!)}x^{2n}-\frac{1\cdot 3\cdots(2n-1)}{2^n(n!)}x^{2n+2}+\cdots)
\end{aligned}
$$

(b) 令 a_n 表示 $f(x)$ 之 MacLaurin 級數中 x^n 項之係數, 則

$$
a_n = \begin{cases} \frac{1\cdot 3\cdots(2m-1)}{2^m(m!)}, & \text{若 } n=2m+1, m=0,1,2,\cdots \\ -\frac{1\cdot 3\cdots(2m-1)}{2^m(m!)}, & \text{若 } n=2m+2, m=0,1,2,\cdots \end{cases}
$$

因 $a_n = \frac{f^{(n)}(0)}{n!}$, 故 $f^{(n)}(0) = a_n(n!)$, 因此

$$
f^{(n)}(0) = \begin{cases} \frac{1\cdot 3\cdots(2m-1)}{2^m(m!)}[(2m+1)!], & \text{若 } n=2m+1, m=0,1,2,\cdots \\ -\frac{1\cdot 3\cdots(2m-1)}{2^m(m!)}[(2m+2)!], & \text{若 } n=2m+2, m=0,1,2,\cdots \end{cases}
$$