1092 Calculus B 01-03 Final Exam

June 24, 2021

There are SIX questions in this examination.

1. Consider the differential equation $y\dfrac{dy}{dt} = \sqrt{1-y^2}$.
   (a) (6 pts) If the constant function $y(t) = a$ is a solution of the equation, find the value of a.
   (b) (10 pts) Given that $y(0) = \dfrac{1}{2}$, solve $y(t)$ for t near 0.

   考慮微分方程式 $y\dfrac{dy}{dt} = \sqrt{1-y^2}$。
   (a) (6 pts) 如果常數函數 $y(t) = a$ 是方程式的解，求a 的值。
   (b) (10 pts) 給定 $y(0) = \dfrac{1}{2}$，求 $y(t)$， 當t在0附近。

2. (16 pts) Solve

$$\begin{cases} y'(t) = t(t^2 - y(t)), \\ y(0) = 3. \end{cases}$$

   (16 pts) 解

$$\begin{cases} y'(t) = t(t^2 - y(t)), \\ y(0) = 3. \end{cases}$$

3. (8 pts) Suppose that X and Y are independent and $\mathrm{E}(X) = 1$, $\mathrm{E}(Y) = 2$, $\mathrm{Var}(X) = 5$, $\mathrm{Var}(Y) = 7$. Compute $\mathrm{E}((2X-Y)^2)$.
   (8 pts) 已知 X，Y爲兩個獨立的隨機變數，而且 $\mathrm{E}(X) = 1$，$\mathrm{E}(Y) = 2$， $\mathrm{Var}(X) = 5$， $\mathrm{Var}(Y) = 7$。求 $\mathrm{E}((2X-Y)^2)$。

4. Suppose that X is a random variable with probability density function $f_X(t) = ce^{\frac{-t^2+8t}{20}}$ for some constant $c > 0$.
   (a) (8 pts) Find the constant c such that $\int_{-\infty}^{\infty} f_X(t)dt = 1$. (Hint: $\int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}$)
   (b) (8 pts) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.
   (c) (6 pts) Suppose that $X_1, \cdots, X_n$ are independent random variables and $X_i \sim X$ for $1 \le i \le n$. Use Chebyshev's inequality to estimate the size of n such that we can derive

$$P\left(\left| \frac{X_1 + \cdots + X_n}{n} - \mathrm{E}(X) \right| < 0.05 \right) > 0.9.$$

   已知隨機變數 X 的機率密度函數爲 $f_X(t) = ce^{\frac{-t^2+8t}{20}}$，其中 $c > 0$ 是一個常數。
   (a) (8 pts) 求常數 c 使得 $\int_{-\infty}^{\infty} f_X(t)dt = 1$. (提示: $\int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}$)
   (b) (8 pts) 求 $\mathrm{E}(X)$ 和 $\mathrm{Var}(X)$.
   (c) (6 pts) 假設 $X_1, \cdots, X_n$ 爲互相獨立的隨機變數而且 $X_i \sim X$， $1 \le i \le n$。 利用 Chebyshev 不等式，估計 n 需要多大，我們才能保證

$$P\left(\left| \frac{X_1 + \cdots + X_n}{n} - \mathrm{E}(X) \right| < 0.05 \right) > 0.9.$$

5. Let $f(t) = \begin{cases} c \cdot te^{-t}, & t > 0 \\ 0, & t \le 0. \end{cases}$

(a) (8 pts) Find constant c such that $f(t)$ is a probability density function.

(b) (4 pts) Suppose that X and Y are independent with probability density functions $f_X = f_Y = f$. Let $Z = X + Y$. Find the plane region D in the xy-plane such that the distribution function of Z, $F_Z(t) = P(X + Y \le t)$, is $\iint_D f_X(x) f_Y(y) dxdy$.

(c) (10 pts) Find the probability density function of Z in part (b).

令 $f(t) = \begin{cases} c \cdot te^{-t}, & t > 0 \\ 0, & t \le 0. \end{cases}$

(a) (8 pts) 求常數 c 使得 $f(t)$ 是一個機率密度函數。

(b) (4 pts) 已知隨機變數 X 和 Y 是獨立的而且它們的機率密度函數爲 $f_X = f_Y = f$。令 $Z = X + Y$。求 xy 平面的區域D，使得 Z的分配函數， $F_Z(t) = P(X+Y \le t)$， 是重積分 $\iint_D f_X(x) f_Y(y) dxdy$。

(c) (10 pts) 求(b)小題中 Z的機率密度函數。

6. The number of flaws in a carpet appears to be Poisson distributed at a rate of one every 6m^2.

(a) (8 pts) Find the probability that a 3m by 4m carpet contains more than 2 flaws.

(b) (8 pts) There are two carpets with sizes 2m by 4m and 2m by 5m. Find the probability that these two carpets together contain less than 2 flaws.

假設地毯上的瑕疵個數呈 Poisson 分配，而且平均每 6平方公尺有一個瑕疵。

(a) (8 pts) 求一條 3公尺寬 4公尺長的地毯有超過 2個瑕疵的機率。

(b) (8 pts) 有兩條地毯，各是 2公尺寬 4公尺長和 2公尺寬 5公尺長。求這兩條地毯上總共的瑕疵數少於 2的機率。