1. (17%) A rumor spreads across a company of 400 employees. 10 employees have heard about it at 9am, and 100 employees at 12pm. Let $y(t)$ be the fraction of employees who have heard about it t hours after 9am. Assume that the rumors spreads at a speed proportional to $y(1-y)$.
   (a) (10%) Write down the differential equation for $y(t)$ and find the solution.
   (b) (3%) How many employees would have heard about the rumor at 3pm?
   (c) (4%) Find y'' at 3pm. Is the rumor spreading speed increasing or decreasing at 3pm?

2. (12%) Solve the differential equation

$$\begin{cases} y'(t) = \dfrac{ty^2(t) - 3y^2(t)}{t^3}, & t > 0 \\ y(1) = -2. \end{cases}$$

3. (12%) Solve the differential equation

$$\begin{cases} (t+1)y'(t) - 2(t^2+t)y(t) = \dfrac{e^{t^2}}{t+1}, & t > -1 \\ y(0) = 5. \end{cases}$$

4. (16%) Let $y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots$ be the solution of the differential equation

$$\begin{cases} \dfrac{dy}{dt} = y - t, \\ y(0) = 2 \end{cases}$$

   by using Taylor's method.
   (a) (6%) Determine the values of a_0, a_1, a_2, a_3.
   (b) (4%) Use the Taylor polynomial $a_0 + a_1 t + a_2 t^2 + a_3 t^3$ to approximate the value of $y(0.2)$. Round your answer to the nearest thousandths (3 decimals).
   (c) (6%) Let $(t_0, y_0) = (0, 2)$ and $h = \Delta t = 0.1$. Use Euler method to get the values y_1 and y_2.

5. (14%) Given $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$, find the following integrals.
   (a) (7%) $\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx$.
   (b) (7%) $\int_{-\infty}^{\infty} e^{-\frac{(x-1)^2}{4}}\,dx$.

6. (17%)
   (a) (7%) Determine the constant α such that

$$f_X(t) = \begin{cases} \alpha t^{-\frac{1}{2}} e^{-t}, & t > 0 \\ 0, & t \le 0 \end{cases}.$$

   is a probability density function.
   (b) (10%) For the random variable $X \sim f_X$, calculate the expectation $\mathbf{E}(X)$ and variance $\mathbf{Var}(X)$ of X.

7. (12%) $f_X(t) = \dfrac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} C$, $t \in \mathbb{R}$ is the probability density function for a random variable X where μ and σ are constants and $\sigma > 0$. Define the random variable Y by $Y = \dfrac{X-\mu}{\sigma}$. Follow the steps below and determine the probability density function $f_Y(s)$ for Y.
   (a) (3%) Explain why $\mathbf{P}(Y \le s) = \mathbf{P}(X \le \mu + \sigma s)$
   (b) (3%) By the above statement, we have the equality of two integration as $\int_{-\infty}^{s} f_Y(y)\,dy =$ ________________.
   (c) (6%) Assume that we can use fundamental theorem of calculus in this case, determine $f_Y(s)$.