1. (20%) Solve the following differential equations.
   (a) (10%) $y'(t) = (\tan t)(y^2 - 1)$, $y(0) = 2$, for t near 0.
   (b) (10%) $t^2 y'(t) - 2ty(t) = 3t^{\frac{3}{2}}$, $y(1) = 0$ for $t > 0$.

2. (18%) Suppose that the number of fruit flies in a test tube is $P(t)$ on day t and $P(t)$ satisfies the differential equation $P'(t) = \lambda P(t)(M - P(t))$, where $\lambda, M > 0$ and M is the capacity.
   (a) (8%) Given that $P(t_0) = P_0$ for some $t_0 > 0$ and $0 < P_0 < M$, solve $P(t)$. (Please show your work.)
   (b) (4%) If the population growth rate, $P'(t)$, obtains maximum value at $P = \tilde{P}$, find $\tilde{P}$ in terms of M.
   (c) (6%) Researchers observe that the population growth rate achieves maximum when the number of fruit flies is 50, and the population size increases from 50 to 70 two days later. Estimate the population size if you wait for another two days. (Round to the nearest whole number.)

3. (10%) Let X be a random variable such that

$$P_X(-1) = \frac{1}{5}, \quad P_X(0) = \frac{1}{5}, \quad P_X(1) = \frac{2}{5}, \quad P_X(2) = \frac{1}{5}.$$

   Let Y be the random variable defined by the equation $Y = X^2 + 3$.
   (a) (4%) Find possible values of Y and the probability function of Y.
   (b) (6%) Calculate $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$.

4. (14%) Toss a fair coin for 100 times. Let Z be the number of heads.
   (a) (6%) Find $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$.
   (b) (4%) Write down Chebyshev's Inequality.
   (c) (4%) Estimate $P(35 \le Z \le 65)$ with Chebyshev's Inequality.

5. (10%) X is a random variable with probability density function $f_X(t) = \frac{1}{\sqrt{\pi}} e^{-t^2}$. Let $Z = X^2$.
   (a) (4%) Write down the distribution function of Z. (You don't need to evaluate the integral.)
   (b) (6%) Find the probability density function of Z.

6. (12%)
   (a) (5%) Suppose that random variables X and Y are independent and $P_X(i) = \frac{m^i}{i!} e^{-m}$, $P_Y(j) = \frac{n^j}{j!} e^{-n}$, for $i, j = 0, 1, 2, \cdots$, $m, n > 0$. Let $Z = X + Y$, show that $P_Z(k) = \frac{(m+n)^k}{k!} e^{-(m+n)}$.
   (b) (7%) Given that the number of accidents in different rush hours periods form independent Poisson distributions and the average is 1 for rush hours in the morning and is 2 in the evening. Find the probability for having 3 or more accidents in one day's rush hours (morning and evening included).

7. (16%) Suppose that the time required for different cashiers form independent exponential distributions. The average time is 2 minutes for cashier A and 4 minutes for cashier B.
   (a) (6%) Cashier A has just started to serve a customer and another customer is waiting in line. If you also decide to wait in line for cashier A, the waiting time has the probability density function $f_X(t) = \frac{1}{4} t e^{-\frac{1}{2}t}$, for $t > 0$. Find the expected waiting time.
   (b) (10%) (Continued) Cashier B has just started to serve a customer and no one is in line. Find the probability that you will be served earlier if you switch from A to B.